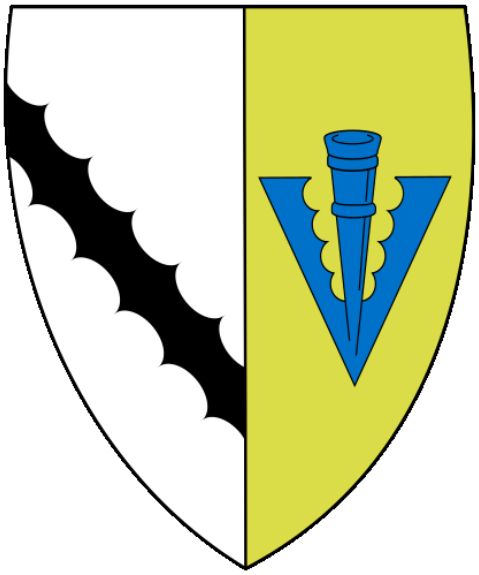
Max Carroll

Type Error Debugging in Hazel

Computer Science Tripos, Part II



Sidney Sussex College College
University of Cambridge
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Declaration of Originality

I, the candidate for Part II of the Computer Science Tripos with Blind Grading Number 2328E, hereby declare that this report and the work described in it are my own work, unaided except as may be specified below, and that the report does not contain material that has already been used to any substantial extent for a comparable purpose. In preparation of this report, I adhered to the Department of Computer Science and Technology AI Policy. I am content for my report to be made available to the students and staff of the University. Date 12/05/2025

Proforma

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Original Aims of the Project

This project seeks to explore ways to improve *type error* debugging. Traces to dynamic type errors can provide better intuition to *why* static type errors were found. Additionally, dynamic errors do not often point directly back to source code, or do so only incompletely. This project aimed to create a source code highlighting system for dynamic errors and an automated search procedure for dynamic errors. The Hazel language was chosen, allowing the project to explore both static and dynamic errors and their interaction. Only a subset of Hazel was expected to be supported, but enough to demonstrate the methods' promise.

Work Completed

The project successfully supports *almost all* of Hazel. Hazel is an intensely active research project so significant work went into staying up to date and producing a corpus of Hazel programs. Four searching methods were implemented, the best failing to find existing dynamic errors for only 2% of the corpus. Dynamic error highlighting had no prescribed basis to build upon. Therefore, a very significant part was devising *novel* formalisations for type-directed highlighting systems which propagate information throughout evaluation; further extended to explain *static* errors also. Both error highlighting implementations were found to be concise and demonstrated to be effective.

Special Difficulties

None

[1] *Words in text calculated by* `texcount`*. Including: tables and footnotes. Excluding: the front matter, bibliography, and appendices*

[2] *Git diff, between* `Evaluation` *branch and* `dev` *branch. Includes* `.ml`*,* `.re`*, dune files, and* `.sh` *files. Excludes* `/evaluaton/data/*`

Contents

Introduction

Software bugs are an inherent part of programming, often leading to unexpected behaviour and system failures. Debugging these errors is a *time-consuming* process taking between 20-60% of active work time [26], with programmers spending a *highly skewed* proportion of their time identifying and resolving a small proportion of *difficult* bugs [19].

Type systems aim to alleviate some of this burden by classifying expressions and operations that are allowed to work on them. This may be done *statically* at compile time or *dynamically* during runtime. The expressions not conforming to the type system manifest themselves as *type errors*.

In static typing, blame for type errors are typically localised to a single location in the code. However, this localisation may be misleading, as the actual cause of the error might be rooted in a broader context, for example in OCaml 65% of type errors related to *multiple* locations [28]. Additionally, the errors only *state* the expected types, but with no explanation for *why*.

In dynamic typing, type errors are found later only appearing during runtime with specific inputs. Additionally, they don't generally specify any source code context which caused them. However, such an error is accompanied by an evaluation trace, which can be *more intuitive* [44], demonstrating concretely why values are ill-typed programs go wrong.

Aims: This project seeks to improve user understanding of type errors by explaining static type errors more *completely*, and *combining* the benefits of static and dynamic type errors. I consider three directions to achieve this, implementing three features to achieve them for use in the Hazel language [2]:

1. Can we explain *static type errors* more *completely*, highlighting the code that determines *why* the error expects it's inconsistent type?

   Being more *complete*, this would alleviate the issue of static errors being incorrectly localised, while also helping build understanding of *why* the errors occur.

   Solution: I devise a *novel* method: **type slicing**. Including formal mathematical foundations built upon the formal *Hazel calculus* [24]. Additionally, it generalises to highlight all code relevant to typing any expressions (not just errors).

2. Can we track source code which contributes to a *dynamic type error*?

   This would provide missing source code context to understand how types involved in a dynamic type error originate from the source code.

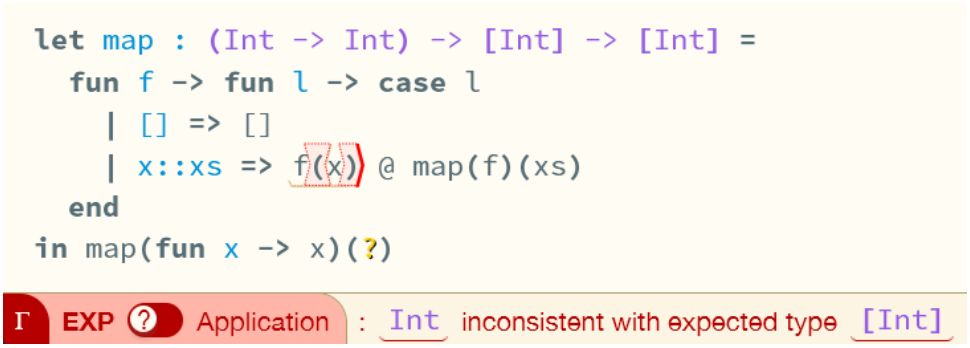
   Solution: I devise a *novel* method: **cast slicing**. Also having formal mathematical foundations. Additionally, it generalises to highlight source code relevant to requiring any specific runtime casts.

3. Can we provide dynamic evaluation traces to explain *static type errors*?

   This would provide an *intuitive* concrete explanation for static type errors.

   Solution: I implement a **type error witness search procedure**, which discovers inputs (witnesses) to expressions which cause a *dynamic type error*. This is based on research by Seidel et al. [32] who devised and evidenced the usefulness of a similar procedure for a subset of OCaml.

Hazel [2] is a functional, locally inferred, and gradually typed research language that allows writing *incomplete programs* under active development at the University of Michigan. Being gradually typed, it is a natural choice for this project, allowing both static and dynamic code to coexist. I successfully demonstrate the utility of these three features in improving understanding of *both* static and dynamic errors as well as how the two classes of errors interact and may be linked automatically.



Note: ? represents a hole, an *incomplete* part of the program.

Figure 1.1: Example: Concatenation `@` has been used instead of cons `::`, but `f(x)` is incorrectly localised as the error. Type slicing will give a full context highlighting the terms involved in the error (the annotation, `f(x)` and `@` operator). The search procedure could find a witness (e.g. `[0]`) concretely evaluating to a dynamic error. The cast in the dynamic error will contain a slice highlighting the source code that makes `@` take lists as input. The results are demonstrated in the evaluation section 4.9.2

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1.1. Related Work

There has been extensive research into the field of programming languages and debugging, attempting to understand *what* is needed [38], *how* developers fix bugs [21], and a plethora of compiler improvements and tools. This project builds adds to this body of research in new ways, focusing on the Hazel language which is itself an active research project being taken in various directions but of particular note as a *teaching language* [20] for students; these features can additionally help with building understanding of bidirectional type systems.

To my knowledge the ideas of *type slicing* and *cast slicing* are novel. However, they were inspired, but differing substantially, to *program slices*, originally explored by Weiser [78], slices in expression-based languages [43], *dynamic program slicing* [74] and *type error slicing* [58, 55], which similarly relate to type systems.

The *type witness search procedure* is based upon Seidel et al. [32], but with significant differences, which will be explained throughout.

Preparation

In this chapter I present the technical background knowledge for this project: an introduction to the type theory for understanding Hazel's core semantics, an overview of Hazel implementation, and notes on non-determinism. Following this, I present my software engineering methodology.

2.1. Background Knowledge

2.1.1. Type Systems

A *type system* is a lightweight formal mathematical method which categorises values into *types* and expressions into types that evaluate to values of the same type. It is effectively a static *approximation* to the runtime behaviour of a language. The following sections expect basic knowledge formal methods of type systems in terms of judgements (appendix A reviews this). Note that I will use *partial functions* to represent typing assumption contexts.

Dynamic Type Systems

Dynamic typing has purported strengths allowing rapid development and flexibility, evidenced by their popularity [51, 13]. Of particular relevance to this project, execution traces are known to help provide insight to errors [44], yet statically typed languages remove the ability to execute programs with type errors, whereas dynamically typed languages do not.

A *dynamically typed system* can be implemented and represented semantically by use of dynamic *type tags* and a *dynamic type* [71]. Then, runtime values can have their type checked at runtime and *cast* between types. This suggests a way to encode dynamic typing via *first-class*[1] cast expressions which maintain and enforce runtime type constraints alongside a dynamic type written `?`.

Cast expressions can be represented in the syntax of expression by $e\langle\tau_1 \Rightarrow \tau_2\rangle$ for expression e and types τ_1, τ_2, encoding that e has type τ_1 and is cast to new type τ_2. An intuitive way to think about these is to consider two classes of casts:

- *Injections* – Casts *to* the dynamic type $e\langle\tau \Rightarrow ?\rangle$. These are effectively equivalent to type tags, they say that e has type τ but that it should be treated dynamically.
- *Projections* – Casts *from* the dynamic type $e\langle? \Rightarrow \tau\rangle$. These are type requirements, for example the add operator could require inputs to be of type `Int`, and such a projection would force any dynamic value input to be cast to `Int`.

Then *injections* meeting *projections*, $v\langle\tau_1 \Rightarrow ? \Rightarrow \tau_2\rangle$ represent an attempt to perform a cast $\langle\tau_1 \Rightarrow \tau_2\rangle$ on v. We check the cast is valid and perform if so:

$$\frac{\tau_1 \text{ is castable to } \tau_2}{v\langle\tau_1 \Rightarrow ? \Rightarrow \tau_2\rangle \mapsto v'} \qquad \frac{\tau_1 \text{ is not castable to } \tau_2}{v\langle\tau_1 \Rightarrow ? \Rightarrow \tau_2\rangle \mapsto v\langle\tau_1 \Rightarrow ? \not\Rightarrow \tau_2\rangle}$$

Compound type casts will be decomposed during evaluation. For example, applying v to a function wrapped in a cast decomposes the cast into casting the applied argument and then the result:

$$(f\langle\tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle)(v) \mapsto (f(v\langle\tau_1' \Rightarrow \tau_1\rangle)\langle\tau_2 \Rightarrow \tau_2'\rangle)$$

Or if f has the dynamic type:

$$(f\langle? \Rightarrow \tau_1' \to \tau_2'\rangle)(v) \mapsto (f(v\langle\tau_1' \Rightarrow ?\rangle)\langle? \Rightarrow \tau_2'\rangle)$$

Hence, casts around functions (type information) will be moved to the actual arguments at runtime, meeting with casts casts on the argument, resulting in a cast error or a successful cast.

[1] Directly represented in the language syntax as expressions.

Gradual Type Systems

A *gradual type system* [34, 52] combines static and dynamic typing. Terms may be annotated as dynamic, marking regions of code omitted from type-checking but still *interoperable* with static code. For example, the following type checks:

```
let x : ? = 10;  // Dynamically typed
x ++ "str"       // Statically typed
```

Where `++` is string concatenation expecting inputs to be `string`. But would then cause a runtime *cast error* when attempting to calculate `10 ++ "str"`.

It does this by representing casts as expressed previously. The language is split into two parts:

- The *external language* – where static type checking is performed which allows annotating expressions with the dynamic type.
- The *internal language* – where evaluation and runtime type checking is performed via cast expressions.[2] The example above would reduce to a *cast error*:

$$10\langle \texttt{Int} \Rightarrow ? \not\Rightarrow \texttt{String} \rangle \texttt{ ++ "str"}$$

For type checking, a *consistency* relation $\tau_1 \sim \tau_2$ is introduced. This is a weakening of the type equality requirements in normal static type checking, allowing *consistent* types to be used instead. Where every type τ is consistent with the dynamic type $?$.

$$\frac{}{\tau \sim ?} \qquad \frac{}{\tau \sim \tau} \qquad \frac{\tau_1 \sim \tau_2}{\tau_2 \sim \tau_1} \qquad \frac{\tau_1 \sim \tau_1' \quad \tau_2 \sim \tau_2}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

Then typing rules can be written to use consistency instead of equality. For example, application typing:

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2' \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \qquad \tau_2 \sim \tau_2'}{\Gamma \vdash e_1(e_2) : \tau_2'}$$

Where $\blacktriangleright_\to$ extracts the argument and return types from a function type, used to account for if $\Gamma \vdash e_1 : ?$, where we treat $?$ then as a dynamic function $? \blacktriangleright_\to ? \to ?$. Intuitively, $e_1(e_2)$ has type τ_2' if e_1 has type $\tau_1' \to \tau_2'$ or $?$ (then treated as $? \to ?$), and e_2 has type τ_1 which is consistent with τ_1' and hence is assumed that it can be passed into the function.

But, for evaluation to work the static type information needs to be encoded into casts to be used in the dynamic internal language, for which the evaluation semantics are defined. This is done via *elaboration*, similarly to Harper and Stone's approach to defining (globally inferred) Standard ML [61] by elaboration to an explicitly typed internal language XML [68]. The *elaboration judgement* $\Gamma \vdash e \rightsquigarrow d : \tau$ reads as: external expression e is elaborated to internal expression d with type τ under typing context Γ. For example, to insert casts around function applications:

$$\frac{\Gamma \vdash e_1 \rightsquigarrow d_1 : \tau_1 \qquad \Gamma \vdash e_2 \rightsquigarrow d_2 : \tau_2' \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \qquad \tau_2 \sim \tau_2'}{\Gamma \vdash e_1(e_2) : \tau \rightsquigarrow (d_1\langle \tau_1 \Rightarrow \tau_2 \to \tau \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) : \tau}$$

If e_1 elaborates to d_1 with type $\tau_1 \sim \tau_2 \to \tau$ and e_2 elaborates to τ_2' with $\tau_2' \sim \tau_2$ then we place a cast[3] on the function d_1 to $\tau_2 \to \tau$ and on the argument d_2 to the function's expected argument type τ_2 to perform runtime type checking of arguments. Intuitively, casts must be inserted whenever type consistency is used, but deciding which casts to insert is non-trivial [29].

The runtime semantics of the internal expression is that of the *dynamic type system* discussed above (2.1.1). A cast is determined to succeed iff the types are *consistent*.

[2] i.e. the proposed *dynamic type system* above.

[3] This cast is required, as if $\tau_1 = ?$ then we need a cast to realise that it is even a function. Otherwise $\tau_1 = \tau_2 \to \tau$ and the cast is redundant.

Bidirectional Type Systems

A *bidirectional type system* [36] takes on a more algorithmic definition of typing judgements, being more intuitive to implement. They also allow some amount of local type inference [62].

This is done in a similar way to annotating logic programs [79, p. 123], by specifying the *mode* of the type parameter in a typing judgement, distinguishing when it is an *input* (type checking) and when it is an *output* (type synthesis).

We express this with two judgements:

$$\Gamma \vdash e \Rightarrow \tau$$

Read as: *e synthesises a type* τ *under typing context* Γ. Type τ is an *output*.

$$\Gamma \vdash e \Leftarrow \tau$$

Read as: *e analyses against a type* τ *under typing context* Γ. Type τ is an *input*

When designing such a system care must be taken to ensure *mode correctness* [76]. Mode correctness ensures that input-output dataflow is consistent such that an input never needs to be *guessed*. For example the following function application rule is *not* mode correct:

$$\frac{\Gamma \vdash e_1 \Leftarrow \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 \Leftarrow \tau_1}{\Gamma \vdash e_1(e_2) \Leftarrow \tau_2}$$

We try to *check* e_2 with input τ_1 which is *not known* from either an *output* of any premise nor from the *input* to the conclusion, τ_2. On the other hand, the following *is* mode correct:

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \to \tau_2 \qquad \Gamma \vdash e_2 \Leftarrow \tau_1}{\Gamma \vdash e_1(e_2) \Leftarrow \tau_2}$$

Where τ_1 is now known, being *synthesised* from the premise $\Gamma \vdash e_1 \Rightarrow \tau_1 \to \tau_2$. As before, τ_2 is known as it is an input in the conclusion $\Gamma \vdash e_1(e_2) \Leftarrow \tau_2$.

Such languages will have three obvious rules. That variables can synthesise their type, being accessible from the typing assumptions. Annotated terms synthesise their type from the annotation (after checking the validity). Subsumption: a synthesising term successfully checks against that same type.

2.1.2. The Hazel Calculus

Hazel is a language that allows the writing of incomplete programs, evaluating them, and evaluating around static and dynamic errors.

It does this via adding *holes*, which can both be typed and have evaluation proceed around them seamlessly. Errors can be placed in holes allowing continued evaluation.

The core calculus [24] is a gradually and bidirectionally typed lambda calculus. Therefore it has a locally inferred bidirectional *external language* with the dynamic type ? elaborated to an explicitly typed *internal language* including cast expressions.

The full semantics are documented in the Hazel Formal Semantics appendix B, with only rules relevant *holes* discussed in this section. The combination of gradual and bidirectional typing system is itself non-trivial, but only particularly notable consequences are mentioned here.[4] The intuition should be clear from the previous gradual and bidirectional typing sections.

Syntax

The syntax, in Fig. 2.1, consists of *types* τ including the dynamic type ?, *external expressions* e including (optional) annotations, *internal expressions* d including cast expressions.

Notating $(\!|)^u$ or $(\!|e|\!)^u$ for empty and non-empty holes respectively, where u is the *metavariable* or name for a hole. Internal expression holes, $(\!|)^u_\sigma$ or $(\!|e|\!)^u_\sigma$, also maintain an environment σ mapping variables x to internal expressions d. These internal holes act as *closures*, recording which variables have been substituted during evaluation.[5]

[4] The difficulties combining gradual and bidirectional typing are largely orthogonal to adding holes.

[5] This is required, as holes may later be substituted with terms containing variables, receiving their values from the closure environment.

$$\tau ::= b \mid \tau \to \tau \mid ?$$
$$e ::= c \mid x \mid \lambda x : \tau.e \mid \lambda x.e \mid e(e) \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid e : \tau$$
$$d ::= c \mid x \mid \lambda x : \tau d \mid d(d) \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow ? \not\Rightarrow \tau \rangle$$

Figure 2.1: Syntax: *types* τ, *external expressions* e, *internal expressions* d. With x ranging over variables, u over hole names, σ over $x \to d$ *internal language* substitutions/environments, b over base types and c over constants.

External Language

Holes synthesise the *dynamic type*, a natural choice made possible by the use of gradual types:

$$\text{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow ?} \qquad \text{SEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow ?}$$

One notable consequence of combining gradual and bidirectional typing is that the *subsumption rule* in bidirectional typing is naturally extended to allow subsuming any terms of *consistent* types:

$$\text{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau}$$

Of course, e should type check against τ if it can synthesise a consistent type. The goal of type consistency is to allow type treating terms as if they were of any consistent type.

Internal Language

The internal language is explicitly typed with typing judgement, $\Delta; \Gamma \vdash d : \tau$. Where Δ is a *hole context*, mapping each hole *metavariable* u to it's *checked type* τ[6] and it's type context Γ under which the hole was typed.

Elaboration

Cast insertion is performed by elaborating to the *internal language*, and must also output an additional context for holes: $\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta$ and $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$.

The resulting hole context will record each hole's original *analysing* type along with the typing assumptions for its hole closure. Recording them instead as $?$ would lose type information.

A well-typed external expression elaborates to a well-typed internal expression *consistent* with the external type.

Final Forms

The primary addition of Hazel is the addition of a new kind of *final forms* and *values*. This is what allows evaluation to proceed around holes and errors. There are three types of final forms:

- *Values* – Constants and functions.
- *Boxed Values* – Values wrapped in casts which cannot be further reduced.
- *Indeterminate Final Forms* – Terms containing holes that cannot be directly evaluated, e.g. holes or function applications where the function is indeterminate, e.g. $(\!|\,|\!)^u(1)$.

Importantly, *any* final form can be treated as a value (in a *call-by-value* context). For example, they can be passed inside a (determinate) function: $(\lambda x.x)((\!|\,|\!)^u)$ can evaluate to $(\!|\,|\!)^u$.

[6] As originally required when typing the external language expression.

Dynamics

A small-step contextual dynamics [30, ch. 5] is defined on the internal expressions to define a *call-by-value* evaluation order, values in this sense are *final forms*.

Like the *refined criteria* [34], Hazel presents a rather different cast semantics designed around *ground types*, that is, base types (`Bool` etc.) and least precise compound types, e.g. $\texttt{Bool} \to \texttt{Bool} \blacktriangleright_{\text{ground}} \texttt{?} \to \texttt{?}$. This formalisation more closely represents common dynamically typed language implementations which only use generic type tags like *fun*, corresponding to the ground type $\texttt{?} \to \texttt{?}$. However, the idea of type consistency checking when *injections* meet *projections* remains the same, with projections/injections now being to/fro *ground types*.

Hole Substitutions

Holes are indexed by *metavariables* u, and can hence also be substituted. Hole substitution is a *meta* action $[\![d/u]\!]d'$ meaning substituting each hole named u for expression d in some term d' with the holes environment. Importantly, the substitutions d can contain variables, whose values are found by checking the holes *environment*, effectively making a *delayed substitution*. See the following rule:

$$[\![d/u]\!](\!|)^u_\sigma = [[\![d/u]\!]\sigma]d$$

When substituting a matching hole u, we replace it with d and apply substitutions from the environment σ of u to d, after first substituting any occurrences of u in the hole's environment σ. This corresponds to *contextual substitution* in contextual modal type theory [48].

This can be thought of as a *fill-and-resume* functionality, allowing incomplete program parts to be filled during evaluation rather than only before evaluation.

2.1.3. The Hazel Implementation

The Hazel implementation [3] is written primarily in ReasonML and OCaml with approximately 65,000 lines of code using advanced OCaml features: GADTs, functors, various preprocessors [39, ch. 9, 10, 25], and custom let operators [14]. It is under very active development, with much of the code being undocumented; this dissertation refers to the implementation as of April 2025.

Language Features

Hazel implements the afore mentioned core calculus along with many relevant additional features:[7]

- **Lists** – Linked lists, in the style of ML. By use of the dynamic type, Hazel can represent *heterogeneous* lists, which may have elements of differing types.
- **Tuples & Labelled Tuples**[8] – Allowing compound terms to be constructed and typed [57, ch. 11.7-8].
- **Sum Types** – Representing a value as one of many labelled variants, each of possibly different types [57, ch. 11.10].
- **Type Aliases** – Binding a name to a type, used to improve code readability or simplify complex type definitions.
- **Pattern Matching** – Checks a value against a pattern and destructures it accordingly, binding its sub-structures to variable names.
- **Explicit Polymorphism** – System F style parametric polymorphism [57, ch. 23]. Where explicit type functions bind arbitrary types to names, which may then be used in annotations. Polymorphic functions are then applied to a type, uniformly giving the corresponding monomorphic function. Implicit and ad-hoc polymorphic functions can still be written as dynamic code, without use of type functions, but are untyped.

[7] Additionally, Hazel supports other features, which do not concern this project.
[8] Merged towards the end of the project's development.

- **Iso-Recursive Types** – Types defined in terms of themselves, allowing the representation of data with potentially infinite or *self-referential* shape [57, ch. 22-23], for example linked lists or trees.

2.1.4. Bounded Non-Determinism

Input generation for a witness search procedure [32] is a *non-deterministic* algorithm [80]. At a high level, non-determinism can be represented declaratively by two ideas:

- *Choice* (`<||>`): Determines the search space, flipping a coin will return heads *or* tails.
- *Failure* (`fail`): The *empty* result, no solutions to the algorithm.

Suppose the non-deterministic result of the algorithm has type τ. These can be represented by operations:

$$\texttt{<||>} : \tau \to \tau \to \tau$$

$$\texttt{fail} : \tau$$

Where `<||>` should be *associative* and `fail` should be a *zero element*, forming a *monoid*:

$$x \texttt{ <||> } (y \texttt{ <||> } z) = (x \texttt{ <||> } y) \texttt{ <||> } z \qquad \texttt{fail <||> } x = x = x \texttt{ <||> fail}$$

That is, the order of making binary choices does not matter, and there is no reason to choose failure.

There are many proposed ways to represent and manage non-deterministic programs which I considered, of which a monadic representation over a tree based state-space model was chosen as a good balance of flexibility, simplicity, and familiarity to other Hazel developers. Appendix H reasons and details other options considered: continuations, effect handlers, tagless final DSLs, direct implementation.

Monadic Non-determinism: Some monads can be extended to represent non-determinism by adding the `choice` and `fail` operators satisfying the usual laws. These operations interact by `bind` distributing over `choice`, and `fail` being a left-identity for `bind`.

$$\texttt{bind}(m_1 \texttt{ <||> } m_2)(f) = \texttt{bind}(m_1)(f) \texttt{ <||> bind}(m_2)(f)$$

$$\texttt{bind}(\texttt{fail})(f) = \texttt{fail}$$

In this context, bind can be thought of as *conjunction*: if we can map each guess to another set of choices, `bind` will conjoin all the choices from every guess. Figure 2.2 demonstrates how flipping a coin followed by rolling a dice can be conjoined, yielding the choice of all pairs of coin flip and dice roll.

Distributivity represents this interpretation: guessing over a combined choice is the same as guessing over each individual choice and then combining the results. `fail` being the left identity of `bind` states that you cannot make any guesses from the no choice (`fail`).

```
let coin = return(Heads) <||> return(Tails);
let dice = return(1) <||> ... <||> return(6);
let m = coin
    >>= flip => // Flip a coin
        dice
    >>= roll => // Roll a dice
        return((flip, roll)) // Return conjunction
```

Figure 2.2: Examples: Bind (`>>=`) as Conjunction

2.2. Starting Point

Concepts

Only the basic foundations of most concepts in understanding Hazel were covered in Part IB Semantics of Programming (and Part II Types later). The concept of gradual typing briefly appeared in Part IB Concepts of Programming Languages, but was not formalised. Monads and non-determinism were also present in this course, but not their intersection.

Tools and Source Code

My only experience in OCaml was from the Part IA Foundations of Computer Science course. This project builds directly upon the open-source Hazel language codebase [3]. The type witness search procedure is inspired by Seidel et al. [32], however my implementation differs significantly, being applied to Hazel rather than OCaml. Three (DFS, BFS, BDFS) searching methods for monadic non-determinism are based on Spivey [53] with minor changes.

2.3. Requirements Analysis

The motivation for this project was a desire to assist programmers by improving type error debugging. Hazel was naturally as we can both static and dynamic aspects together. To achieve this, three features were to be implemented:

1. A more complete highlighting system for static type errors.
2. A (complete) source code highlighting system for dynamic errors.
3. Automated discovery of dynamic witnesses for static errors.

The first aim is not present in the project proposal, but naturally followed from the direction that the theorising of cast slicing took.

The first two features required novel solutions and theory. After this design, a set of core goals and extensions were specified. Most extensions relate to *maintainability* of code, *conciseness* of slices, witness *coverage* improvements, and *usability* (UI improvements). The goal was to *devise* and *implement* the features to an extent that proves their potential, with less focus on user interface.

Core Goals: Create/translate a corpus of ill-typed Hazel programs for evaluation usage. The witness search procedure must have reasonable coverage ($> 75\%$) in a time suitable for interactive debugging (30s) over the corpus. Implement synthesis and analysis type slice theories (section 3.1.4, 3.1.5), and cast slice theory (section 3.2) including basic UI highlighting of the source code.

Extensions: Implement contribution slice theory (section 3.1.6). Allow customisable witness instantiation ordering. *(Maintainability)* Segregate type slicing logic from type checking semantics. Segregate cast slicing logic from transition semantics. *(Conciseness)* Error slices (*). Minimised error slices (**).

Low-Priority Extensions: *(Coverage)* Extended pattern instantiation (**). *(Usability)* Trace visualisation & compression. Graphs for cast dependence. UI to select sub-parts of slices (*).

Marked extensions were added after the 1st (*) or 2nd (**) evaluation.

Each goal and extension must be achieved for a sufficient subset of Hazel, with features classified according to the MoSCoW method [35] in fig. 2.3.

2.4. Software Engineering Tools and Techniques

Methodology: Due to the high level of uncertainty and risk in devising new theories, a spiral development model was followed. Each iteration refined the implementation through defined milestones (see project proposal) and repeated evaluation (fig. 2.4). Further extensions were prioritised based on their risk of scheduling-overruns.

Class	Features
Must Implement	Base types: their constants & operations, lists, functions, bindings, type aliases, tuples, sum types & constructors, holes, casts.
Should Implement	Pattern matching
Could Implement	labelled tuples (*), type functions, recursive types.
Won't Implement	Tests, deferrals, probes (*), filters, live literals.

(*) New features later merged from main branch not present in Hazel when the project proposal was written.

Figure 2.3: Hazel Subsets to Implement for

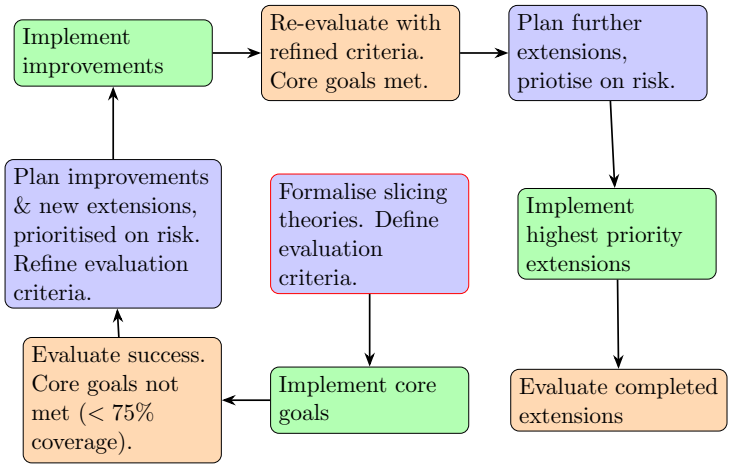


Figure 2.4: Phases of development

Hazel Codebase & Interaction: The Hazel codebase is extensive (65k lines), with much of it being undocumented. As such, interaction with the Hazel development team was required to clarify workings. Equally, I found and raised issues on bugs throughout, some being fixed by myself and later merged into the main development branch. I re-use their existing build, formatting, and deployment systems and conform with their coding standards.

Version Control & Merges: Git and GitHub were used for version control and backups. Hazel is a very active research project, so new features were added over the course of developing this project. These updates were regularly merged into my project, often requiring extensive conflict resolution. 18 hours were spent entirely on merges, see appendix I for a list.

Continuous Integration & Deployment: The main branch of this project is integrated with Hazel's continuous integration and deployment system. Therefore, the main branch of this project can be accessed at https://hazel.org/build/witnesses-type-slicing/.[9]

Testing: Hazel performs it's testing by manually listing code samples with errors labelled by comments. Or, more recently, shifting towards unit regression tests using the *Alcotest* package [1]. I reuse these to ensure type checking and evaluation remains correct. However, as slicing involves random term IDs, for this I use the earlier method of testing directly within the editor.

Tools: No new dependencies were introduced into Hazel, instead existing dependencies were used, e.g. `Js_of_ocaml` [5] for regex matching, Jane Street `Base.Sequence` [4] for streams. However, micro-benchmarking of the search procedure was performed using *Bechamel* [10].

Licences: Hazel is open-source available under an MIT licence. The source OCaml corpus of ill-typed programs [27] translated into Hazel is freely available under a Creative Commons Zero (CC0) licence. I again license the project with an MIT licence.

[9] This is continuously deployed. New functionality implemented *after* graduation (04/06/25) may be present in future.

Implementation

This section details the slicing theories developed and implementations of type slicing, cast slicing, and the search procedure.

3.1. Type Slicing Theory

I developed a novel method, *type slicing*, to aid programmers in understanding *how* a bidirectional type system works. First, I define typing slices. Then, three slicing criteria, each associating typing derivations with different explanatory slices.

The first two criteria give insight on the synthesised and analysed type contributions. The third on any code contributing in any way to a term's type.

The second and third criterion were *very challenging* to formalise, requiring non-obvious mathematical machinery: *context typing slices* (section 3.1.2), *checking contexts* (appendix C.4), and *type-indexed slices* (section 3.1.3). Only the basic definitions are given here, the full theory is found in appendix C.

3.1.1. Expression Typing Slices

First, I introduce what *slices* are in this context. The aim is to provide a formal representation of term highlighting.

Term Slices

A *term slice* is a term with some sub-terms omitted. The omitted terms are those that are *not* highlighted. For example if my slicing criterion is to *omit terms which are typed as* `Int`, then the following expressions highlights as:

$$(\lambda\, x : \texttt{Int}\, .\, \lambda y : \texttt{Bool}.\ x\,)(\,1\,)$$

Omitted sub-terms are replaced by a *gap* term, notated $\square$. Representing the example above, we get:

$$(\lambda \square.\ \lambda y : \texttt{Bool}.\ \square)(\square)$$

We can then define a *precision* partial order [82] on term slices: $\varsigma_1 \sqsubseteq \varsigma_2$ meaning ς_1 is less or equally precise than ς_2. That is, ς_1 matches ς_2 structurally except that some sub-terms may be gaps. For example:

$$\square \sqsubseteq \square + \square \sqsubseteq 1 + \square \sqsubseteq 1 + 2$$

Lattice Structure: For any *complete term* t (having no gaps), the slices of t form a *bounded lattice structure* [81]. That is, every pair ς_1, ς_2 has a *join* $\varsigma_1 \sqcup \varsigma_2$ and *meet* $\varsigma_1 \sqcap \varsigma_2$. In general, not all slices slices have joins: $1 \not\sqcup 2$, but do have meets as $\square \sqsubseteq \varsigma$ for all ς.

Typing Assumption Slices

Expression typing is performed given a set of *typing assumptions*. Therefore, in addition, we also desire a slice taking the *relevant* assumptions. Typing assumptions are *partial functions* mapping variables to types (see appendix A).

Hence, their slices are partial functions to *type slices*. Such that, a slice maps no more variables to no more precise types. This, and meets and joins, can be defined pointwise:

Definition 1 (Typing Assumption Slice Precision)**.** *For typing assumption slices* γ_1, γ_2*. Where* $\mathrm{dom}(f)$ *is the set of variables for which a partial function* f *is defined:*

$$\gamma_1 \sqsubseteq \gamma_2 \iff \mathrm{dom}(\gamma_1) \subseteq \mathrm{dom}(\gamma_2) \text{ and } \forall x \in \mathrm{dom}(\gamma_1).\ \gamma_1(x) \sqsubseteq \gamma_2(x)$$

Definition 2 (Typing Assumption Slice Joins and Meets)**.** *For typing slices* γ_1, γ_2*, and any variable* x*:*

If $\gamma_1(x) = \bot$ *then* $(\gamma_1 \sqcup \gamma_2)(x) = \gamma_2(x)$ *and* $(\gamma_1 \sqcap \gamma_2)(x) = \bot$*, analogously if* $\gamma_2(x) = \bot$*. Otherwise,* $(\gamma_1 \sqcup \gamma_2)(x) = \gamma_1(x) \sqcup \gamma_2(x)$*.*

Again, slicing complete typing assumptions Γ forms a bounded lattice. In general, some slices have no join: consider $x : \texttt{Int}$ and $x : \texttt{String}$.

Expression Typing Slices

Finally, an *expression typing slice*, ρ, is a pair, ς^γ, of a term slice and a typing slice. Precision, joins and meets, can be extended componentwise to term typing slices with all the same properties.

Typing: *Expression slices* can be *type checked* under the *type assumption slices* by replacing gaps $\square$ by: holes of any metavariable $(\!|\,|\!)^u$ in *expressions*, fresh variables in *patterns*, and the dynamic type in *types*. Notated by $[\![\cdot]\!]$.

Definition 3 (Expression Typing Slice Type Checking)**.** *For expression typing slice* ς^γ *and type* τ*.* $\gamma \vdash \varsigma \Rightarrow \tau$ *iff* $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Rightarrow \tau$ *and* $\gamma \vdash \varsigma \Leftarrow \tau$ *iff* $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Leftarrow \tau$*.*

3.1.2. Context Typing Slices

Next, some of an expression's type might be enforced by the surrounding *context*. For example, the type of the underlined expression below is enforced by the surrounding highlighted annotation:

$$\underline{(\lambda x.(\!|\,|\!)^u)} : \texttt{Bool} \to \texttt{Int}$$

Contexts and Their Slices

We represent these surrounding contexts by a *term context* $\mathcal{C}$, which marks *exactly one* subterm as $\bigcirc$. Where $\mathcal{C}\{t\}$ substitutes term t for the mark $\bigcirc$ in $\mathcal{C}$, only allowed if the marked position expects a term of the same class as t (pattern `Pat`, type `Typ`, or expression `Exp`). Notate the classes by $\mathcal{C} : \texttt{X} \to \texttt{Y}$. Context composition can be defined as substituting contexts into contexts if their classes match. It is *crucial* that composition is defined *syntactically*, and not pointwise, allowing syntactic deconstruction of contexts (used in AFun rule in appendix fig. C.2).

These extend to context slices notated c analogously to term slices. However, the precision relation $\sqsubseteq$ is more restrictive, requiring the mark $\bigcirc$ to remain in the same structural position. For example: $\bigcirc(\square) \sqsubseteq \bigcirc(1)$, but $\bigcirc \not\sqsubseteq \bigcirc(1)$. Concisely defined pointwise:

Definition 4 (Context Precision)**.** *If* $c : \texttt{X} \to \texttt{Y}$ *and* $c' : \texttt{X} \to \texttt{Y}$ *are context slices, then* $c' \sqsubseteq c$ *if and only if, for all terms* t *of class* `X`*, that* $c'\{t\} \sqsubseteq c\{t\}$*.*

We find that filling contexts preserves the precision relations both on term slices *and* context slices:

Proposition 1 (Context Filling Preserves Precision)**.** *For context slice* $c : \texttt{X} \to \texttt{Y}$ *and term slice* ς *of class* `X`*. Then if we have slices* $\varsigma' \sqsubseteq \varsigma$*,* $c' \sqsubseteq c$ *then also* $c'\{\varsigma'\} \sqsubseteq c\{\varsigma\}$*.*

Joins and meets can be defined pointwise as before, still forming bounded lattices over complete contexts. The lattice bottom is the *purely structural context*, consisting of only gaps with the mark in the correct position. In general, in addition to joins, not all contexts have meets: $\bigcirc \not\sqcap \bigcirc(\square)$.

Typing Assumption Contexts and Their Slices

The accompanying typing notion can be represented by *endomorphisms on typing assumption slices*. These functions represents which *relevant* typing assumptions must be *added*, and those safely *removable* when typing an expression within a context slice.

Precision, joins, and meets can be defined pointwise, forming bounded lattices on complete functions as usual. The bottom element being the constant function to the empty typing assumptions. Again, such functions are monotone:

Proposition 2 (Function Application Preserves Precision)**.** *For typing assumption slice* γ *and typing assumption context slice* f*. Then if we have slices* $\gamma' \sqsubseteq \gamma$*,* $f' \sqsubseteq f$ *then also* $f'(\gamma') \sqsubseteq f(\gamma)$*.*

Context Typing Slices

Finally, an *expression context typing slice*, $\boldsymbol{\rho}$ is a context slice with each sub-context recursively tagged with typing assumption context slices. Lattice relations are defined componentwise. Composition again must be carefully defined *syntactically* (appendix definition 37).

Type Checking: An analogous $[\![\cdot]\!]$ translation can be defined.

3.1.3. Type-Indexed Slices

Decomposing slices by their type is required for cast slicing and useful in calculating slices according to analysis slices and hence also contribution slices. For example consider the following context slice explaining why the underlined term analyses `Bool → Int`:

$$\underline{(\lambda x.(\!|\,)^u)} : \texttt{Bool} \to \texttt{Int}$$

This would be tagged with the type `Bool → Int` where a sub-slice considering *only* the `Int` return type (omitting the `Bool` annotation) can be extracted:

$$\underline{(\lambda x.(\!|\,)^u)} : \texttt{Bool} \to \texttt{Int}$$

This section will only consider *context slices*, but term slices are type-indexed analogously.

The main property that indexed-slices should maintain is that slices can be *reconstructed* from their sub-parts. Joining the sub-slices will produce the full type. As sub-slices may slice different regions of code, we pair them with contexts which place the sub-slices within the same context, making them join-able.

Definition 5 (Type-Indexed Context Typing Slices)**.** *Syntactically defined:*

$$\mathcal{S} ::= p \mid p * \mathcal{S} \to p * \mathcal{S}$$

With any $\mathcal{S}$ only being valid if it has a full slice. The full slice of $\mathcal{S}$, notated $\overline{\mathcal{S}}$, is defined:

$$\overline{p} = p$$

$$\overline{p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2} = p_1 \circ \overline{\mathcal{S}_1} \sqcup p_2 \circ \overline{\mathcal{S}_2}$$

Then left *(incremental)* composition and right *(global)* composition can be defined, by composing at the upper type constructor or at the leaves respectively:

Definition 6 (Type-Indexed Context Typing Slice Composition)**.** *For type-indexed context typing slices $\mathcal{S}$ and $\mathcal{S}'$. If $\mathcal{S} = p$ and $\mathcal{S}' = p'$:*

$$p' \circ p = \overline{p'} \circ \overline{p} \qquad p \circ p' = \overline{p} \circ \overline{p'}$$

If $\mathcal{S} = p$ and $\mathcal{S}' = p'_1 * \mathcal{S}'_1 \to p'_2 * \mathcal{S}'_2$:

$$S \circ \mathcal{S}' = (p \circ p'_1) * \mathcal{S}'_1 \to (p \circ p'_2) * \mathcal{S}'_2$$

If $\mathcal{S} = p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2$:

$$\mathcal{S} \circ \mathcal{S}' = p_1 * (\mathcal{S}_1 \circ \mathcal{S}') \to p_2 * (\mathcal{S}_2 \circ \mathcal{S}')$$

This definition stems from it representing regular context typing slice composition over its full slices.

Proposition 3 (Type-Indexed Composition Preserves Full Slice Composition)**.** *For type-indexed slices $\mathcal{S}$ and $\mathcal{S}'$:*

$$\overline{\mathcal{S} \circ \mathcal{S}'} = \overline{\mathcal{S}} \circ \overline{\mathcal{S}'}$$

Application, notated $|>$, can be defined similarly, converting indexed context slices into valid indexed expression slices. The opposite direction is more difficult and can be found in appendix (appendix C.3.4).

3.1.4. Criterion 1: Synthesis Slices

Synthesis slices aim to explain why an expression *synthesises* a type. They omit all sub-terms which analyse against a type retrieved from synthesising some other part of the program. For example, the following term synthesises a $\texttt{Bool} \to \texttt{Bool}$ type, and the variable $x : \texttt{Int}$ and argument are irrelevant:

$$(\lambda\, x : \texttt{Int}\, .\, \lambda y : \texttt{Bool}.\, y)(\, 1\,)$$

Definition 7 (Synthesis Slices)**.** *For a synthesising expression, $\Gamma \vdash e \Rightarrow \tau$. A synthesis slice is an expression typing slice ς^{γ} of e^{Γ} which also synthesises τ, that is, $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Rightarrow \tau$.*

Proposition 4 (Minimum Synthesis Slices)**.** *A minimum synthesis slice of $\Gamma \vdash e \Rightarrow \tau$, under $\sqsubseteq$, exists and is unique.*

These slices can be calculated via a typing judgement $\Gamma \vdash e \Rightarrow \tau \mid \mathcal{S}$, meaning $\mathcal{S}$ is the minimum type-indexed synthesis slice of e^{Γ} synthesising τ. The judgement rules mimic Hazel's typing rules, giving an algorithm to calculate minimum synthesis slices (see appendix C.5).

3.1.5. Criterion 2: Analysis Slices

A similar idea can be devised for type analysis, represented using *context slices*. After all, it is the terms immediately *around* the sub-term where the type checking is enforced. For example, when checking this annotated term:

$$(\lambda x.\underline{(\!|\,|\!)^{u}}) : \texttt{Bool} \to \texttt{Int}$$

The *inner hole term* $(\!|\,|\!)^{u}$ (underlined) is required to be consistent with $\texttt{Int}$ due to the annotation and lambda constructor present in its context. The analysis slice will be:

$$(\lambda\, x\, .\, \underline{(\!|\,|\!)^{u}}\,) : \texttt{Bool} \to \texttt{Int}$$

In other words, if the context slice was type checked, then the inner hole would *still* be required to analyse against $\texttt{Int}$. However, the overall synthesised type of the whole context may differ, the above would synthesis $\texttt{?} \to \texttt{Int}$ vs. the original $\texttt{Bool} \to \texttt{Int}$.

Checking Context

We only want to consider the smallest context *scope* that enforced the type checking. For example, the below term has 3 annotations, but only the inner one enforces the $\texttt{Int}$ type on the integer 1:

$$\underline{1} : \texttt{Int}\ : \texttt{?} : \texttt{Bool}$$

I refer to this as the *minimally scoped checking context*. Note that checking contexts will always synthesise a type. To give another example, an integer argument's type is enforced by the annotation on the function:

$$(\lambda\, x : \texttt{Int}.\ (\!|\,|\!)^{u}\,)(\,\underline{1}\,)$$

Definition 8 (Checking Context)**.** *For term e checking against τ: $\Gamma \vdash e \Leftarrow \tau$. A checking context for e is an expression context $\mathcal{C}$ and typing assumption context $\mathcal{F}$ such that:*

- $\mathcal{C} \neq \bigcirc$.
- $\mathcal{F}(\Gamma) \vdash \mathcal{C}\{e\} \Rightarrow \tau'$ *for some* τ'.
- *The above derivation has a sub-derivation* $\Gamma \vdash e \Leftarrow \tau$.

Definition 9 (Minimally Scoped Checking Context)**.** *For a derivation $\Gamma \vdash e \Leftarrow \tau$, a minimally scoped expression checking context is a checking context of e such that no sub-context is also a checking context.*

All minimally scoped checking contexts can be constructed syntactically via rules (appendix C.4). For a sub-term in a program, there will be a unique minimally scoped context which matches with the program structure (appendix proposition 25). Analysis slices are slices of minimally scoped checking contexts.

Definition 10 (Analysis Slice)**.** *For $\Gamma \vdash e \Leftarrow \tau$ with a minimally scoped checking context $\mathcal{C}^{\mathcal{F}}$. An analysis slice is a context slice $c^{\mathcal{f}}$ of $\mathcal{C}^{\mathcal{F}}$ where $[\![c^{\mathcal{f}}]\!]$ is also a checking context for e.*

Conjecture 1 (Minimum Analysis Slices)**.** *A minimum analysis slice of $\Gamma \vdash e \Leftarrow \tau$ in a checking context $\mathcal{C}^{\mathcal{F}}$, under $\sqsubseteq$, exists and is unique.*

This can again be calculated by a judgement reading as, *e which type checks against τ in checking context $\mathcal{C}$ has (type-indexed) analysis slice $\mathcal{S}$*:

$$\Gamma; \mathcal{C} \vdash e \Leftarrow \tau \mid p$$

The rules build upon rules defining minimally scoped checking contexts, and are more complex, making use of *type-indexed* synthesis slices, see fig. 3.1. This requires an involved conversion of (indexed) expression slices to context slices (appendix C.3.4).

$$(\lambda x : ?.\lambda y : \texttt{Int}.y)(\texttt{true})$$

(a) A function: synthesising $\texttt{Int} \to \texttt{Int}$.

$$(\lambda\, x : ?\, .\lambda y : \texttt{Int}.y)(\,\texttt{true}\,)$$

(b) Its synthesis slice.

$$(\lambda\, x : ?\, .\lambda\, y : \texttt{Int}.\, y\,)(\,\texttt{true}\,)$$

(c) The sub-slice relating *only* to the input part Int.

$$(\lambda\, x : ?\, .\lambda\, y : \texttt{Int}.\, y\,)(\,\texttt{true}\,)(\,\underline{1}\,)$$

(d) The analysis slice of the function's argument ($\underline{1}$) when applied. Uses the synthesis sub-slice from **(c)**.

Figure 3.1: Demonstration: Analysis slice application uses synthesis slices

3.1.6. Criterion 3: Contribution Slices

This criterion highlights all regions of code which *contribute* to typing succeeding. That is, all sub-terms who could change their type to make the term ill-typed. For example, for the underlined term:

$$(\lambda f : \texttt{Int} \to ?.\ f(1))\underline{(\lambda x : \texttt{Int}.\ x)}$$

Both contextual and synthetic parts (in dark yellow) contribute:

$$(\lambda\, f : \texttt{Int} \to ?\, .\ f(1)\,)\ \underline{(\lambda\, x : \texttt{Int}.\ x\,)}$$

Notice that sub-terms which do not contribute must be dynamically annotated. Therefore, this criterion omits the dynamic code regions.

These can be calculated mimicking the Hazel typing rules. During subsumption, remove the synthesis slice sub-part which match with dynamic parts of the analysing type. Contextual parts are found by passing context slices directly as inputs to type analysis rules.

3.2. Cast Slicing Theory

Cast slicing propagates type slice information during evaluation, by tagging casts types with type slices. A primary reason in formalising type-indexed slices was to make slices decomposable during evaluation, retaining only sub-slices relevant to the part of the type involved in the decomposed cast. The first two criteria work together during elaboration, analysis slices inserted when casting on a elaborated expression who had it's type *analysed*, and the synthesised for synthesis slices. The rules are very involved, but build directly upon the Hazel calculus elaboration semantics; the rules are found in appendix C.9.

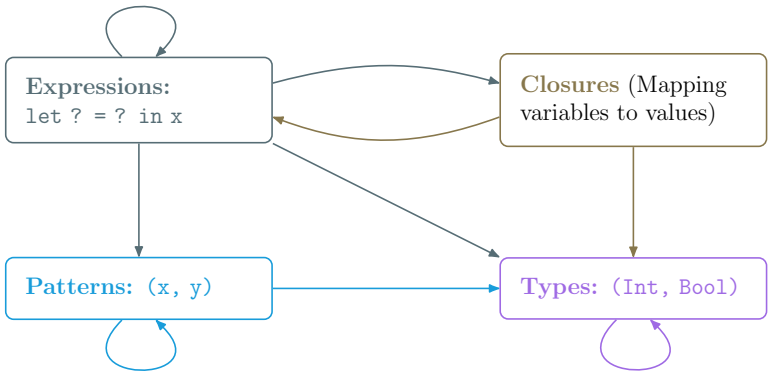


Figure 3.2: Mutually Recursive Hazel Terms.

3.3. Type Slicing Implementation

Here I detail how the theories above were adapted to produce an implementation for Hazel.

3.3.1. Hazel Terms

Hazel represents its terms as a standard abstract syntax tree (AST) via mutual recursion. Every sub-term is tagged with an identifier (ID, `ID.t`). Terms are grouped similarly to the calculus (see section 2.1.2), but combining external and internal expressions, and adding patterns and environments, see fig. 3.2 and 3.3.

```
let (x, y) : (Int, Bool) = (1, true) in x
```

Figure 3.3: Let binding a tuple pattern with a type annotation.

3.3.2. Type Slice Data-Type

Expression slices as ASTs

Storing expression slices directly as ASTs is both *space and time inefficient*, even when accounting for the persistence [65, ch. 2] of trees in OCaml.

For highlighting purposes, there is no need to retain the structure (unlike the theory, we do not need to type-check the results, instead just assuming correctness).

Unstructured Code Slices

With this in mind, I represent slices indirectly by their IDs with an *unstructured* list, referred to now as a *code slice*. Additionally, this allows more *granular* control over slices, as they need not conform with the structure of expressions, which is taken advantage of in reducing slice sizes (section 4.8.1).

Type-Indexed Slices

Cast slicing and contribution slices required *type-indexed* slices. I therefore tag type constructors with slices recursively, i.e.:

```
type typslice_typ_term =
  | Unknown
  | Arrow(slice_t, slice_t)  // Function type
  | ... // Type constructors
and typslice_term = (typslice_typ_term, code_slice)
and typslice_t = IdTagged.t(slice_term)
```

Figure 3.4: Initial Type Slice Data-Type

However, this did not model the structure of type slices particularly well. Analysis slices add ids to *all* sub-slices, giving *linear* space complexity in the depth of the type.

Incremental Slices

Therefore, slices are represented incrementally. With *incremental slices* for synthesis slice parts and *global slices* for analysis slice parts.

A *global slice* is only tagged once, then *lazily* tagged to sub-slices if used. That is, when deconstructing types, e.g. in function matching. We get the type in fig. 3.5:

```
...
and typslice_empty_term = [
  | `Typ(typ_term)
  | `TypSlice(typslice_typ_term)
]
and typslice_incr_term = [
  | `Typ(typ_term)
  | `TypSlice(typslice_typ_term)
  | `SliceIncr(typslice_typ_term, code_slice)
]
and typslice_term = [
  | `Typ(typ_term)
  | `TypSlice(typslice_typ_term)
  | `SliceIncr(typslice_empty_term)
  | `SliceGlobal(typslice_incr_term, code_slice)
]
and typslice_t = IdTagged.t(typslice_term)
...
```

Figure 3.5: The type slice data-type

The invariant that a slice has at most *one* incremental and/or global slice is maintained by splitting into three types (`empty_term`, `incr_term`, `term`). Regular un-sliced types `` `Typ(...) `` are maintained to provide easier interoperability with the rest of the code-base, also allowing type slicing to be turned off.

Polymorphic Variants [39, ch. 7.4], notated `[ | ... ]` are used to more conveniently write functions on slices. This is possible due *row polymorphism* [66] [15, ch. 10.8] relating the variants by a *structural subtyping* relation [73]. We have that:[1]

$$\texttt{typslice\_empty\_term} :> \texttt{typslice\_incr\_term} :> \texttt{typslice\_term}$$

Type constructors are either co-variant or contra-variant [69, ch. 2] with respect to the subtyping relation. For example, id tagging is covariant, so:

$$\texttt{IdTagged.t(typslice\_incr\_term)} :> \texttt{IdTagged.t(typslice\_incr\_term)} = \texttt{typslice\_t}$$

Functions are bifunctors: contravariant in their input and covariant in their output, for example:

$$\texttt{typslice\_incr\_term} \to \texttt{typslice\_incr\_term} :> \texttt{typslice\_empty\_term} \to \texttt{typslice\_term}$$

This function subtyping property significantly reduces work in defining functions on slices (see fig. 3.6).

Utility Functions: Functions on slices often only concern the underlying type, e.g. checking if a slice is a list type. Writing direct pattern matching code on `typ_term` and `typslice_term` is easier. An `apply` function can apply these direct functions to the term inside a slice. The bottom two branches can both be passed into `apply` function as they are sub-types of `typslice_term`. Many other utility functions are implemented, including mapping functions, wrapping functions, unpacking functions, matching functions.

[1] $x :> y$ meaning x a subtype of y.

```
let rec apply = (f_typ, f_slc, s) =>
  switch (s) {
  | `Typ(ty) => f_typ(ty)
  | `TypSlice(slc) => f_slc(slc)
  | `SliceIncr(s, _) => apply(f_typ, f_slc, s)
  | `SliceGlobal(s, _) => apply(f_typ, f_slc, s)
  }
```

Figure 3.6: Apply Utility Function

Type Slice Joins

Type joins (section 3.1.1) are extensively used in the Hazel implementation for *branching statements* and in the theory of contribution slices.

For basic type slices, when the same type information is available from multiple branches, highlighting only one branch is required, but both for contribution slices. See that the 1 in fig. 3.7 is not highlighted (in green). However, we did still need some information from the left (the 0).

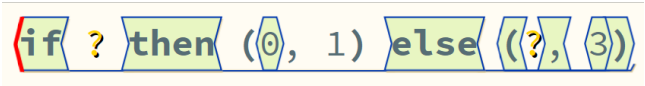


Figure 3.7: Type Slice Joins (note: Hazel syntax uses `?` also for expression holes)

This gets complex, requiring tracking all the data relating to which branches should be highlighted, and accumulating the inconsistent parts of failed joins for use in minimised error slices (demonstrated in section 4.8.1). I define custom OCaml let bindings [14] to retain clarity while automatically combining branches in successful joins and combining inconsistencies in failed joins. For example, see fig. 3.8, where comments describe the logic abstracted by the custom bindings; note that all join-able type combinations use this same parallel binding logic to combine branches and inconsistencies.

3.3.3. Static Type Checking

Hazel is *bidirectionally typed*, where the *mode* (synthesis, analysis) is specified by the `Mode.t` type. Type checking calculates a type information object `Info.t` for each term, stored efficiently in a map from ID keys. `Info.t` is demonstrated in fig. 3.9 with arrows representing dependencies (e.g. a term's type depends on it's mode, self, typing context, and status).

Self and Mode

Slicing logic relating to synthesis slices and analysis slices is factored into `Self.t` and `Mode.t` respectively, cleanly segregated from the type checking code. Doing this still required full understanding of the type checker implementation, ensuring the correct IDs are sliced.

Typing (Co-)Context

The typing context and co-contexts are modified to use type slices. This deviates from the theoretical notion of an expression slice: the structural context in which the variable is used is untracked when passing through the context. Therefore, it requires using *unstructured* code slices. It is useful in practice, allowing slices calculated during binding to be retrieved upon usage, see fig. 3.10.

Status and Type

The status, mode, and self are combined to determine a term's actual type, being dynamic if there is an error. When the expectations (mode, self) are inconsistent, the inconsistent slice information parts are tagged to the error status; section 4.8.1 retrospectively considers what

```
...
| (Arrow(s1, s2), Arrow(s1', s2')) =>
      let+ s1j = join'(s1, s1') // If successful: binds the join to s1j and
                                // propagates the branch used joining s1, s1'
                                // Else: Propagates inconsistencies in s1, s1'
      and+ s2j = join'(s2, s2') // If both successful: binds the join to s2j and
                                // combines previously used branches joining s1, s2'
                                // with those used joining s2 and s2'
                                // Else: propagates inconsistency in s1, s2' (if any)
                                // alongside new inconsistencies in s2, s2' (if any)
      and! branches_used = ();  // If both successful: Binds combined branches
                                // Else: propagates inconsistencies
      ( // If nothing failed, Returns the successful join.
        `SliceIncr((
          Slice(Arrow(s1_join, s2_join)), // The successful join
          choose_branch(branches_used, slice_incr1, slice_incr2),
        )) // Wrap with only one slice: the left if both branches used
        |> temp,
        branches_used, // combined branches used
      );
...
```

Figure 3.8: Joining function types

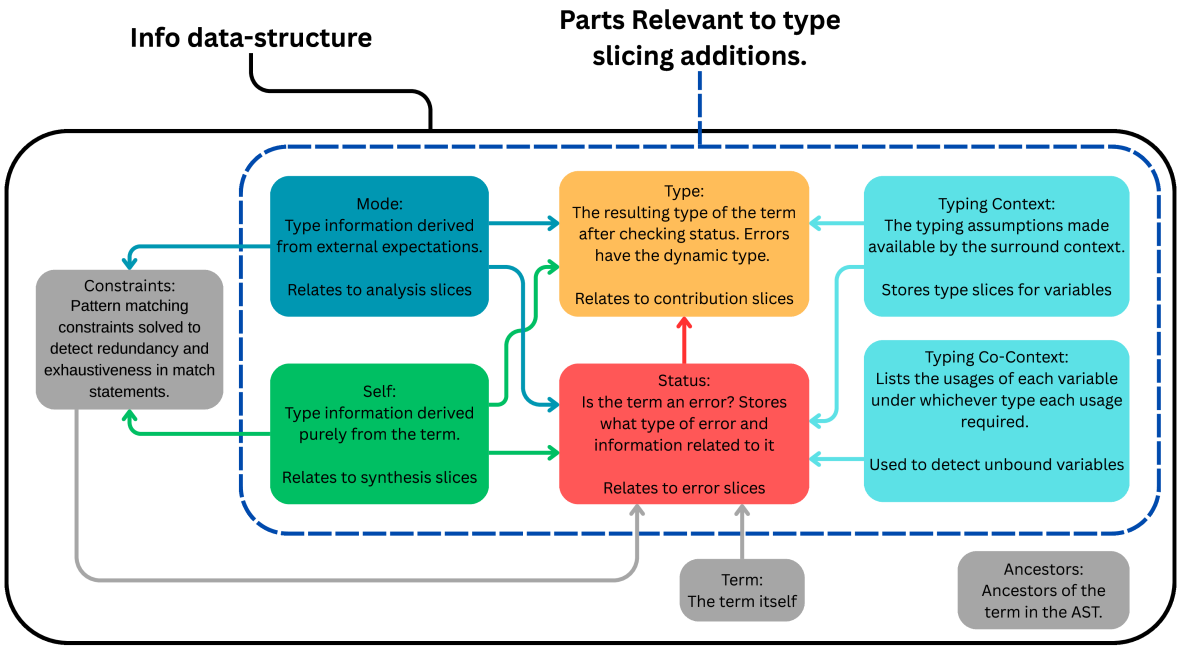


Figure 3.9: `Info.t` data-structure

slice information to be extracted here. Additionally, contribution slices can extract the static parts of synthesis slices here.

3.3.4. Integration

To support the full Hazel language, type slices needed to implement many functions, for example: type substitution, type normalisation, weak-head normalisation, tracking sum types, various structural matching functions etc. Additionally, almost every usage of types in the codebase had to be refactored to use type slices (which are not so easily pattern matched upon) while ensuring slices were correctly maintained.

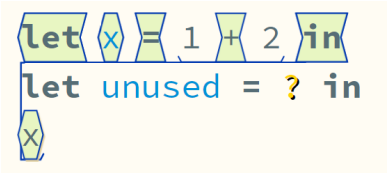
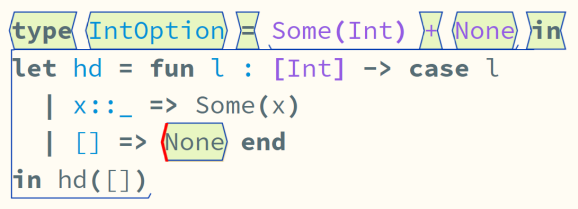


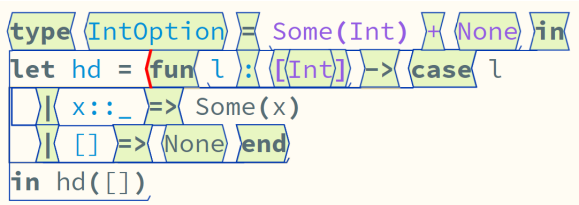
Figure 3.10: Type slice for variable `x`: includes its binding and slice.

3.3.5. User Interface & Examples

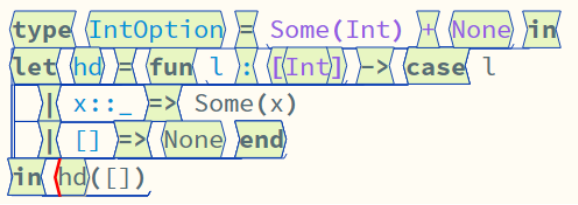
In fig. 3.11, the type slices at the cursor (in red) are highlighted. Error slices distinguish between the synthesis and analysis parts with blue and red, see the evaluation examples section 4.9.2.



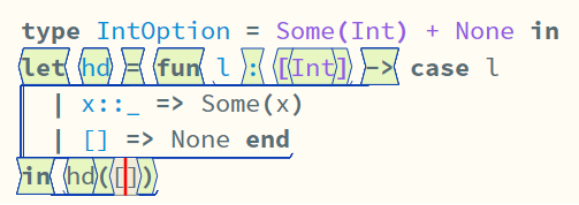
(a) `None` synthesises `IntOption` due to it's type definition.



(b) The function synthesises `[Int]`→ `IntOption` due to its `[Int]` annotation and that the match branches synthesis `IntOption`. Both branches provide the same type information, only one branch (the last) is highlighted.



(c) The variable usage of `hd` synthesise `[Int]`→ `IntOption` similarly, whose slice is retrieved from the typing context.



(d) The list input is expected to be an `[Int]` as it is applied to `hd` which is a function annotated with input type `[Int]`.

Figure 3.11: Type Slices

3.4. Cast Slicing Implementation

To implement cast slicing, replace casts between *types* by casts between *type slices*. Type slices are already type-indexed and retain all type information so can be used equivalently.

3.4.1. Elaboration

Cast insertion recursively traverses the unelaborated term, inserting casts to the term's statically determined type as stored in the `Info` data-structure. For example, a list literal recursively elaborates its terms and joins their slices, casting to the join. Ensuring all the type slice information is retained and/or reconstructed during elaboration was a meticulous and error-prone process.

3.4.2. Cast Transitions

Section 2.1.2 gave an intuitive overview of how casts are treated at runtime. Type-indexed slices allows cast slices to be decomposed in exactly the same way.

However, as Hazel only checks consistency between casts between *ground types*, there are two rules where new[2] casts are *inserted* (ITGround, ITExpand). The new types are created

[2] As opposed to being derived from decomposition.

$$\tau_1 \to \tau_2 \blacktriangleright_{\text{ground}} ? \to ?$$

Figure 3.12: Ground Matching Functions

via a *ground matching* relation taking the topmost compound constructor of types, e.g. ground functions fig. 3.12. Relevant portions of the appendix are fig. B.8, fig. B.10, fig. C.4.

As we store type slices incrementally, the part of the slice corresponding *only* to the outer type constructor is the outer slice tag.

3.4.3. Unboxing

When we know a final form's (section 2.1.2) type, we may need to extract parts of the term according to the type during evaluation. For example, extracting the elements of a tuple. But due to casts and holes, this is not trivial [18].

Slicing does not concern unboxing, but indeterminate evaluation (section 3.5) unveiled bugs within which had to be fixed. I raised and fixed these, with my PR eventually being merged into the main branch. Appendix D details this, after covering the required context on the unboxing implementation.

3.4.4. User Interface & Examples

Type slices within casts can be selected from the evaluation result and displayed. This requires reworking some of the dependencies of Hazel's model-view-update architecture to make sure the cursor has access to the code editor cell when inside the evaluation result cell.

```
let add = fun x -> fun y -> x + y in
add(1)("one")
```

```
≡ 1 + "one" : Int
```

(a) A simple cast error blaming the plus operator for requiring the integer cast.

```
let map = fun f -> fun l ->
  case l
    | [] => []
    | x::xs => f(x) :: map(f)(xs)
  end
in map(fun x : Int -> x)([1,?,3])
```

```
≡ [1: ○, ○: Int: ○, 3: ○]
```

(b) A hole cast to `Int` due to a mapped function annotated with an `Int` input.

```
let f : Int -> Float = float_of_int in
f(?)
```

```
≡ float_of_int(○ : Int)
```

(c) A decomposed cast. The input of the function takes only slice for the argument part `Int` of the function type `Int → Float`.

Figure 3.13: Cast Slicing Examples

3.5. Indeterminate Evaluation

Dynamic errors include evaluation traces, which can aid in debugging [44], yet static type errors lack such traces. Seidel et al. [32] offer an algorithm to search for these traces for static errors in OCaml by lazily, non-deterministically narrowing input holes to least specific values based on their usage context.

This section creates a framework for non-deterministic evaluation of indeterminate expressions by lazily performing hole substitutions using type information from dynamic casts. Unlike

Seidel, this supports more language features (all of Hazel), any number of inputs (holes), and exhaustive generation of these inputs. It is a general evaluation method, not limited to cast error searches. Specifics relating to cast errors and implementing search orderings, are covered in section 3.6.

3.5.1. Resolving Non-determinism

To model infinite non-determinism I create a monadic DSL with an explicitly tree/forest-based representation. The forest model allows for varying low-level search traversals. The module type of combinators is in `Nondeterminism.Search`; its underlying parametric type is `t('a)` with `'a` being the type of the solutions. Section 3.6.2 discusses the actual implementations of this interface, giving four searching procedures.

In addition to the functions from Section 2.1.4, I add standard `map` and `join` functions, and various other functions, e.g. `ifte` modelled after Mercury's if-then-else construct [7].

Abstracting Search Order: Forest Model

Typical stream-based models of non-determinism [75] only admit the possibility of depth-first search (DFS). Stream concatenation provides no way of remembering choice points and backtracking before finishing a computation.

Instead, we can use a model based on forests (lists of trees) [53]. Choice, similarly to streams, is performed by concatenating forests. Finally, to build tree structures, a `wrap` combinator wraps every tree up into a single tree with a new root, see fig. 3.14. Effectively, this encodes a notion of cost for search paths.

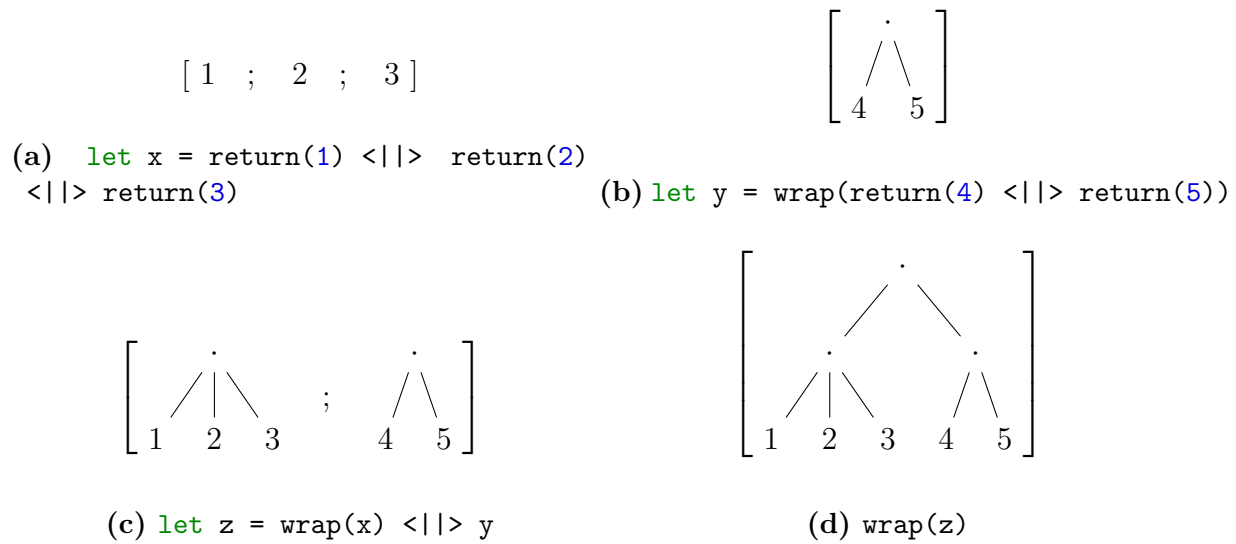


Figure 3.14: Forests Defined Using `wrap`

The DSL is extended with a `wrap : t('a) => t('a)` combinator. As `wrap` is abstract, the underlying implementation does not actually need to use a forest data structure. Therefore, DFS can still be efficiently implemented using regular streams. Finally, a `run` function produces a lazy list of solutions with ordering specified by the search method; these can be cycled through with some arrows in the UI.

Recursive Functions

As OCaml is strict, defining infinite choices via recursion can lead to non-termination during definition. I define a shorthand lazy application function `apply(f, x)` by `return(x) >>= f`, represented infix by `|>-`. Provided that bind lazily applies `f`, recursive functions can be written directly resulting in infinite choices without OCaml's strictness leading to infinite recursion.

3.5.2. A Non-Deterministic Evaluation Algorithm

This section demonstrates one indeterminate evaluation algorithm evaluating terms to concrete values in fig. 3.15 with an accompanying code extract fig. 3.16.

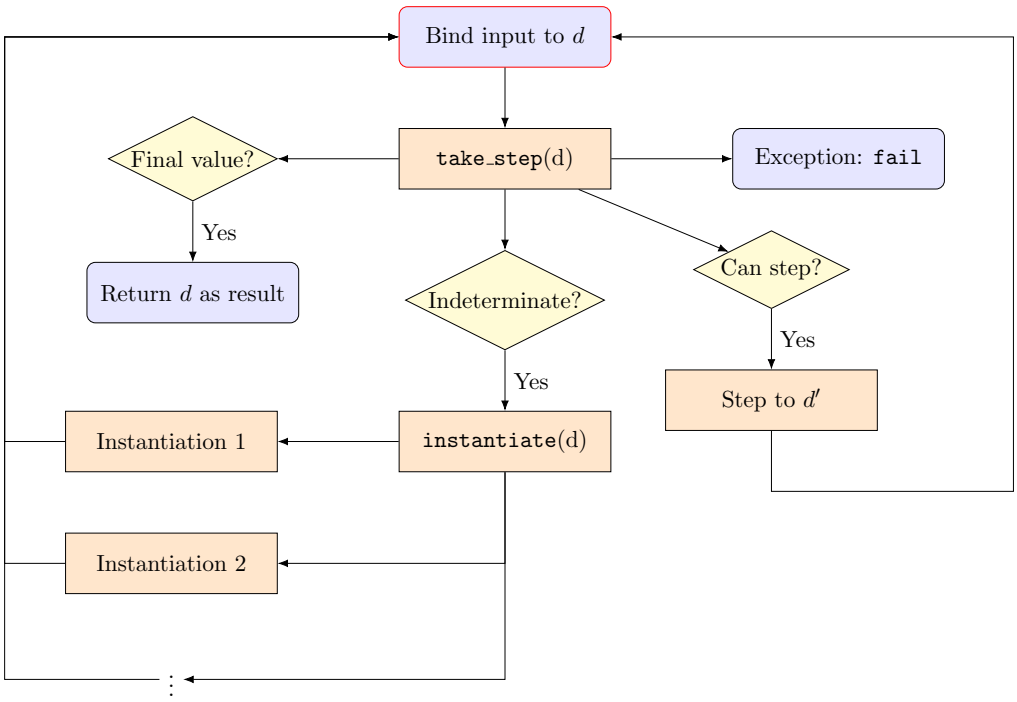


Figure 3.15: Block diagram of indeterminate evaluation to values

Instantiation is implemented by a *non-deterministic* function `instantiate`, discussed in detail in section 3.5.3:

```
Instantiation.instantiate : Exp.t => S.t(Exp.t)
```

Evaluation steps are performed by a (deterministic) function `take_step`, classifying its input as either (concrete) values, indeterminate values, or steppable expressions. The step evaluator re-uses the Hazel stepper logic, which had existing bugs to be fixed: causing non-termination of fixed points[3] and misclassifying concrete values as indeterminate values:

```
OneStepEvaluator.take_step : Exp.t => TryStep.t
```

To ensure that the search tree has finite branching factor, possibly infinite choices must be wrapped, e.g. evaluation steps. The actual implementation additionally threads state tracking number of instantiations and trace length throughout the algorithm.

The generalised indeterminate evaluation takes a higher-order `logic` function which determines the return logic (e.g. returning only results with cast errors).

3.5.3. Hole Instantiation & Substitution

This section details the semantics of hole instantiations, including Hazel-specific issues.

Choosing which Hole to Instantiate

An indeterminate term may contain *multiple* holes or even *no* holes. Which hole needs to be instantiated in order to *make progress*?

When attempting to evaluate the indeterminate term some transitions rules require a subterm to be concrete (e.g. a function during application). We chose to instantiate the hole that blocks the *first* blocked transition rule. If latter holes were instantiated, the term might *still* be unevaluable due to this first hole.

This is implemented using Hazel's evaluator abstraction (`EV_MODE`), which (rather obscurely[4]) separates this logic from the transition semantics. Therefore, hole choosing logic will automatically update to future changes in the transition semantics.

[3] Due to incorrect management of closures; the main branch stepper is still broken as of May 9th.

[4] Lacking documentations and using advanced custom let operators [14]. Understanding took considerable time.

```
module Make = (S: Search) => {
  module Instantiation = Instantiation.Make(S);
  open S;
  open S.Infix;

  let rec values = (d: DHExp.t) : S.t(DHExp.t) => {
    let step = OneStepEvaluator.take_step(d);
    switch (step) {
    | BoxedValue => return(d)
    | Indet =>
      d |>- Instantiation.instantiate
        >>= values;
    | Step(d') => wrap(d' |>- values);
    | exception (EvaluatorError.Exception(_)) => fail
    };
  };
};
```

Figure 3.16: Indeterminate Evaluation to Values

Synthesising Terms for Types

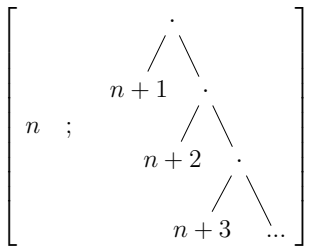
Suppose we know which hole to instantiate and to which type (section 3.5.4). How do we refine these holes fairly and lazily, to the *least specific* value that allows evaluation to continue?

Base types must be instantiated directly to their (possibly infinite set of) values, for example:

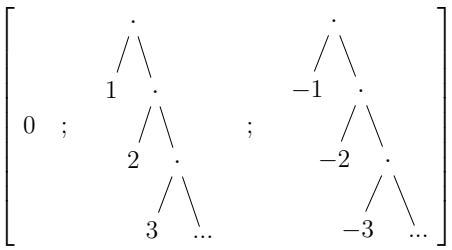
Booleans: `return(true) <||> return(false)`.

Integers: A recursive definition using lazy application `|>-`, see fig. 3.17:

```
let rec ints_from = n => return(n) <||> wrap(n + 1 |>- ints_from)
let nats = ints_from(0)
let negs = ints_from(1) >>| n => -n
let ints = nats <||> wrap(negs)
```



(a) `ints_from(n)`



(b) `ints`

Figure 3.17: Enumerating Integers

Strings: A string is either *empty* or is a string with a first character from a finite set. We can recursively wrap all strings, prefixed by each character. See fig. 3.18:

```
let chars = ... // Choice every single letter string to be considered
let rec strings = () => return("")
  <||> wrap(chars >>= chr =>
            (() |>- strings) >>= str =>
            chr ++ str))
```

Figure 3.18: Enumerating Strings

```
let duplicate = fun x -> (x, x) in
let bound_hole = ? in
duplicate(bound_hole)

≡ (○, ○)
```

Figure 3.19: Duplicated Holes

```
case ?
  | 0 => true
  | 0. => true
  | "zero" => true
  | _ => false
end
```

Figure 3.20: Dynamic/Type-case Match Statement

Other types are *inductive*, these can be represented indirectly by lazily instantiating only their *outermost* constructors:

Lists: A list is either the empty list, [ ] or a cons $?_1 :: ?_2$. To retain the correct dynamic type information, $?_2$ must be cast back to the list type.

Sum Types: Enumerate each of the sum's constructors with their least specific value.

Functions: Constant functions have least specific values $\lambda\_.\ ?$. The function may then be applied to any value, and it's result synthesised after application. This can synthesise any return value, hence errors in the usage of the function will be detected. But, if the *input* has an erroneous type but is not yet caught due to partial annotations, these will be lost. This is rare, and could be mitigated by generating the identity function where possible.

Maintaining Correct Casts: Holes have a dynamic type at runtime. Therefore, the hole's context *expects* a dynamic expression. Therefore, we must cast every instantiation back to the dynamic type.

Substituting Holes

Holes can be bound to variables in the execution environment, and may also be duplicated, before they are required to be instantiated, see fig. 3.19. Every occurrence must be substituted.

Hole substitution was described as part of the Hazel calculus section 2.1.2. But, unexpectedly, the main Hazel branch does not yet implement it. A full implementation of metavariables and delayed closures is complex. Therefore, as hole closures are not required for hole instantiation,[5] I use existing term IDs identify holes, ensuring these are maintained and propagated correctly, so that duplicated holes retain the same ID.[6]

Substitutions must also be performed within closures, eagerly evaluating the results to ensure that closures bind variables to values.

3.5.4. Determining the Types for Hole Substitutions

If we know which hole to instantiate, how do we know which type to instantiate it to? This logic is highly specific to Hazel's cast semantics, with many Hazel-specific issues arising.

For efficiency, my implementation both determines *which hole*, and its *type information* during the same pass.

Directly from Casts: Most of the time, a hole is directly surrounded by a cast, whose type information can be used to perform an instantiation.

Cast Laziness: However, this is *not* always the case. For efficiency reasons, Hazel treats casts over compound data lazily, e.g. casts around tuples will only by pushed inside upon usage of a component of the tuple. Treating casts eagerly is a significant change to the Hazel semantics, so was opted against. Section 4.8.3 evaluates the consequence of this choice.

Pattern Matching: Does a hole even have only one possible type? Dynamic pattern matching actually allows terms to be matched against *non-uniform* types. See fig. 3.20, having patterns of multiple types. Hence instantiating to any of the types might allow progress.

[5] Instantiations do not contain variable references.

[6] All of elaboration and dynamics required these checks.

We can collect each of these possible types from the elaborated casts inserted on the *branches*, non-deterministically rewrapping them around the scrutinee.

Extended Match Expression Instantiation (Pattern Instantiation)
An interesting extension was partially implemented which improves code coverage and additionally detects errors within patterns. It instantiates holes in a match expression according also to the *structure* of each pattern, allowing the instantiation to prioritise searching along each branch. We instantiate the scrutinee with the least specific versions which match the patterns on each branch, e.g. `?::?` for `x::xs`. However, difficulties come when the scrutinee is a compound term, or when least specific matches only indeterminately match earlier branches. Further details addressing these issues can be found in appendix E.

3.6. Search Procedure

Now that a framework for indeterminate evaluation has been specified, the DSL implementations and cast error search logic can be addressed. The error search procedure can be switched on via a toggle in the settings.

3.6.1. Detecting Relevant Cast Errors

To search for cast errors, we must first define what one is. A reasonable definition is those terms which contain *cast failures*; in Hazel, these are casts between *inconsistent ground types*. However, this has some issues:

Multiple Cast Failures: Terms may have multiple cast failures, some of which discovered during static type checking and inserted via elaboration. But these failures will not stop evaluation until necessary. Therefore, we should consider only the casts which are directly *causing* a term to get stuck, this is implemented similarly to choosing which hole to instantiate (section 3.5.3).

Cast Laziness: Only casts between *ground* types are checked for consistency. Due to cast laziness (section 3.5.4), some compound terms will be cast between inconsistent types, but *not* placed within a *cast failure*. As before, this issue is ignored due to requiring large changes to Hazel semantics, the evaluation finds it to be a rare occurrence.

Dynamic Match Statements: When matching dynamically on values with different types, the instantiations wrap the scrutinee in casts to each type. If any of these casts failed, they should not count as witnesses, as they were introduced entirely by the instantiation procedure.

3.6.2. Searching Methods

I implement *four* different search methods, implementing the non-determinism signature specified in section 3.5.1.

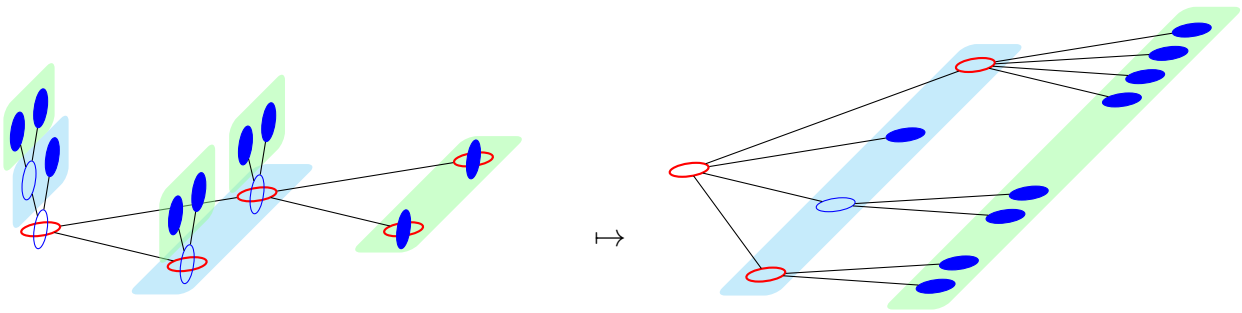
Depth First Search (DFS): Modelling DFS by streams is a typical method [75]: implementing choice and conjunction via appending, and `wrap` being identity function (no internal tree structure needed).

Breadth First Search (BFS): Breadth first search represents forests by sequences of sequences [63], with the nth inner sequences representing every solution in the nth level of the trees in the forest. Then, `choice` concatenates each level, and `wrap` conses the empty list, pushing every solution one level down.

The other monadic operators are complex, but `join : t(t('a)) => t('a)` can be visualised in fig. 3.21: for simplicity showing joining trees of trees, which is extended to forests by choicing the trees in each inner forest.

This join and mapping can be defined lazily; the standard definition for `bind` from the paper [53] does not work easily in OCaml due to its strictness.

Bounded Depth First Search (BDFS): BFS is fair, avoiding non-termination (for finite branching factor), but it has *exponential* space complexity in the depth of the level being explored [31]. In comparison, DFS only requires space linear in the depth explored.



(a) Before Join: Singleton forest of singleton forests (list of lists of lists of lists). Rectangles are the inner lists of the forests.

(b) After Join: Flattened into a single forest (list of lists) by rotating vertical trees into horizontal plane. Solid circles are the solutions actually stored in the BFS lists.

Figure 3.21: BFS Join

Bounded DFS, performs successive depth-bounded depth first searches, retaining low space complexity of DFS while avoiding unfairness. Previous solutions are repeatedly explored, but this does not increase the already exponential time complexity of the search.

Spivey [53] represent this by functions `Int => ['a, Int]`, calculating solutions within an integer depth bound. Solutions are tagged with their remaining depth budgets, which allows keeping only the fringe solutions (zero budget) upon each iteration, so the same solution will not appear twice.

I augment this into an `Bound.t => (Bool, ['a, Bound.t])`, implemented as a functor allowing customised depth bound increases. The boolean is false only if no search path reaches the depth bound, in which case the trees have been fully searched and the algorithm can be terminated.

Interleaved Streams (IDFS): Streams, but with choice and conjunction via interleaving also ensures fairness. This works even for trees with infinite branching factor. However, interleaving has a linear space complexity, resulting in exponential space complexity as with BFS.

3.7. Repository Overview

Branches: The submitted code contains the three most important branches of this project (`Evaluation`, `witnesses-type-slicing`, `contribution-slices`). The `Evaluation` branch factors out a `Js_of_ocaml` [5] dependency from the Hazel core library, meaning evaluation code can be run natively and on the command line; however, the web interface is therefore unusable on this branch.

Hazel Architecture: The Hazel architecture demonstrated in fig. 3.23 with green files/directories added, orange files modified *significantly*, and blue files modified *insignificantly* (but still needing to know how they work) or unmodified. 97 files were added/edited, not all can fit, so anything omitted is considered insignificant to understanding this project. Figure 3.22 shows the structure of the evaluation modules.

Dir	Description	
data/*	Program corpus, pre-filtering	
results/*	Results	
⊢	./benchmarks.file	Results of search procedure benchmarks
⊢	./results.file	Results of effectiveness analysis
⊢	./failure_classification.txt	Classification of failed searches in BDFS
⊢	./log.file	Log from which results are derived
ParseData.ml	Evaluation string data loaded into `.ml` file	
CastSliceUtil.ml	Cast slice size calculation and errors classification	
SlicingUtil.ml	Type slice size calculation and errors classification	
ResourceLimits.ml	Unix alarm-based evaluation timeout	
Settings.ml	Ctx to type check & evaluate under	
Util.ml	Maths utilities	
CastSliceUtil.ml	Cast slice size calculation and errors classification	
Results.ml	Calculates results log	

Figure 3.22: Evaluation Data, Results, and Modules

Dir	Description			
src/*	Hazel source code			
⊢	./haz3lcore/*	Core semantics library		
⊢		./dynamics/*	Hazel dynamics	
⊢			./indeteval/*	Indeterminate Evaluation
⊢				./Nondeterminism.re — Non-determinism DSL & implementations
⊢				./Searching.re — Collection of default search procedures
⊢				./RedexHoleType.re — Chooses hole to instantiate and its possible types
⊢				./CastErrorChecker.re
⊢				./OneStepEvaluator.re
⊢				./Instantiation.re — Instantiation Logic & Hole Substitution
⊢				./IndetEvaluation.re — Generic indeterminate evaluation algorithm
⊢			./state/*	./IndetEvaluatorState.re — IndetEval State
⊢			./stepper/*	Hazel stepper (Bug fixes required)
⊢			./evaluator.re	Big-step semantics: evaluator stack machine
⊢			./transition/*	Transition semantics & Unboxing
⊢				./Casts.re — Cast calculus: modified sum type semantics
⊢				./PatternMatch.re — Pattern matching
⊢				./Transition.re — Small-step semantics
⊢				./Unboxing.re — Unboxing: Various bug fixes
⊢		./lang/term/*	Hazel AST definitions & fundamental functions	
⊢			./Grammar.re	Adds type slices; replace types with slices
⊢			./IdTagged.re	ID tagging terms
⊢			./TypSlice.re	Type Slices (and its utility function)
⊢			./Exp.re	Expressions
⊢			./Typ.re	Types: Added branch tracking logic for joins
⊢			./Pat.re	Patterns
⊢			./TPat.re	Type Patterns
⊢		./prog/*	Settings & Program results (now returning lists)	
⊢		./statics/*	Hazel statics	
⊢			./ConstructorMap.re	Sum constructors: Adds Constructor joining logic
⊢			./Ctx.re	Typing assumptions context
⊢			./CoCtx.re	Typing assumptions co-context
⊢			./Coverage.re	Pattern matching exhaustivity & redundancy checks
⊢			./Self.re	Expectation independent type info: Synthesis slice logic
⊢			./Mode.re	Expectation based type info: Analysis slice logic
⊢			./Info.re	Statics information: Error slices
⊢			./Statics.re	Type Checker: Bindings slicing logic
⊢			./Elaborator.re	Elaboration: Casts are now between slices
⊢		./tiles/*	Structure editor tiles	
⊢		./zipper/*	Structure editor zipper and parser	
⊢		./Joins.re	Slice joins utilities	
⊢	./haz3lmenhrir/*	LR(1) Parser		
⊢	./haz3lweb/*	MVU Web Interface: see `docs/ui-architecture.md`		
⊢		./app	./editors/*	Threads cursor into `./result/*` editors
⊢			⊢	./decoration/Deco.re — Adds UI for slices and error slices
⊢		./Settings.re	Adds indet eval and search settings	
⊢		./www/*	`.css` files: Adds slice colourings	
⊢		./util/*	./WorkerServer.re	Live evaluator: Now uses BDFS indet eval.
⊢	./pretty/*	Formatting for Hazel code		
⊢	./util/*	Utility modules (adds `Base.Sequence`)		
test/*	Test cases: Updates to use type slices			
⊢	./Test_Unboxing.re	Adds list unboxing tests (also merged into `dev` branch)		

Key: Additions, Significant Modifications, Insignificant or No Modifications

Figure 3.23: Hazel Architecture & Additions

Evaluation

This section evaluates how successfully and effectively the implemented features achieve the goals stated in the introduction. Supplementary results are found in appendix F.

4.1. Success

As demonstrated below, the type witness search procedure and slicing features exceeds all core goals (section 2.3). All extensions except for the *usability* extensions were attempted, with extended pattern instantiation (section 3.5.4) only partially implemented.

These were all completed with respect to most must/should/could have Hazel features, except for: type slicing of type functions and labelled tuples,[1] which remain only partially functional.

4.2. Goals

This project devised and implemented three features: *type slicing, cast slicing,* and a *static type error witness search procedure*. Each of which had a clear intention for it's use:

Type Slicing: Expected to give a greater static context to expressions. Explaining why an expression was given a specific type.

Cast Slicing: Expected to provide static context to runtime type casts and propagate this throughout evaluation. Explaining where a cast originated and why it was inserted.

Search Procedure: Finds dynamic type errors (cast errors) automatically, linked back to source code by their execution trace and *cast slice*. Therefore, a static type error can be associated automatically with a concrete dynamic type error *witness* to better explain.

4.3. Methodology

I evaluate the features and their various implementations (where applicable) along *four* axes. Quantitative measures were evaluated over a corpus of ill-typed and dynamically-typed Hazel programs (section 4.5):

Quantitative Analysis

Performance: *Are the features performant enough for use in interactive debugging? Which implementations perform best?*

The time and space usage of the search procedure implementations were micro-benchmarked for each ill-typed program in the corpus. Up to 100 runs were taken per program with estimated time, major and minor heap allocations were estimated using an ordinary linear regression (OLS) via the *bechamel* library [10].

Effectiveness: *Do the features effectively solve the problems? Are the results easily interpretable by a user?*

The *coverage*, what proportion of programs admit a witness, for each search procedure implementation was measured. The search procedure does not always terminate, a 30s time limit was chosen. The coverage was expected to be *reasonable*, chosen at 75%.

Additionally, the *size* of witnesses, evaluation traces, type slices, and cast slices were measured. The intention being that a smaller size implies that there is less information for a user to parse, and hence easier to interpret.[2]

Qualitative Analysis

Critical: *What classes of programs are missed by the search procedure? What are the implications of the quantitative results? What improvements were, or could be made in response to these?*

[1] Labelled tuples were only merged late in the development cycle.

[2] Not necessarily *always* true, but a reasonable assumption.

This section provides *critical* arguments on usefulness or effectiveness, which are *evidenced* by quantitative data. Differing implementations and *subsequent improvements* are compared. Additionally, further unimplemented improvements are proposed.

Holistic: *Do the features work well together to provide a helpful debugging experience? Is the user interface intuitive?*

Various program examples are given, demonstrating how all three features can be used together to debug a type error.

4.4. Hypotheses

I made several hypotheses and investigated each. The evidence and implications being discussed in the following sections.

Search method space requirements: The space requirements for DFS and Bounded DFS are expected to be lower than that of BFS and interleaved DFS.

Type Slices are larger than Cast Slices in Errors: Casts are deconstructed during elaboration and evaluation, so cast slices are expected to be smaller than the original type slices, and therefore more directly explain why errors occur.

The Small Scope Hypothesis: This hypothesis [40] states that a high proportion of errors can be found by generating only *small* inputs. Evidence that this hypothesis holds has been provided for Java data-structures [56] and answer-set programs [42]. Does it also hold for finding dynamic type errors from small *hole instantiations*?

Smaller instantiations correlate with smaller traces: As functional programs are often written recursively, destructuring compound data types on each step. If this and the small scope hypothesis hold, then most errors could be found with *small execution traces*.

4.5. Program Corpus Collection

A corpus of small and mostly ill-typed programs were produced, containing both dynamic (unannotated) programs and annotated programs (containing statically caught errors). We have made this corpus available on GitHub [11].

4.5.1. Methodology

There are no extensive existing corpora of Hazel programs, nor ill-typed Hazel programs. Therefore, we opted to transpile parts of an existing OCaml corpus collected by Seidel and Jhala [27], which is freely available under a Creative Commons CC0 licence.

I am grateful for my supervisor who created a best-effort OCaml to Hazel transpiler [12]. This translates the OCaml examples into both a dynamic example, and a (possibly partially) statically typed version according to what type the OCaml type checker expects expression to be.

This corpus contains both OCaml unification and constructor errors. When translated to Hazel, these may manifest as differing errors. The only errors that the search procedure is expected to detect are those which contain *inconsistent expectations* errors. Hence, the search procedure is ran on the corpus of annotated programs filtering those without this class of errors. Additionally, the search procedure requires the erroneous functions to have holes applied to start the search, these are inserted automatically by the evaluation code after type checking the programs.

4.5.2. Statistics

The program corpus contains **698** small programs, **294** were annotations, of which **203** were applicable to performing the search procedure on.

Ratios vs. DFS	Implementations BDFS	IDFS	BFS
Time	8.3	52	230
Major Heap	9.0	3.2	270
Minor Heap	9.7	83	390

Figure 4.1: Benchmarks: Performance ratios to DFS over common programs

4.6. Performance Analysis

4.6.1. Slicing

The type and cast slicing mechanisms do not increase the time complexity of the type checker nor evaluator. Hence, they are still as performant as the original, capable of interactive use for medium sized programs.

4.6.2. Search Procedure

Only the annotated ill-typed corpus containing inconsistency errors are used in evaluating the search procedure. After all, any well-typed program cannot have a dynamic type error.

As the search procedure may be non-terminating, these results are found given a 30s time limit. Micro-benchmarking the programs which do not time-out, the time and space used searching for each witness can be estimated.

The performance ratios between each implementation as compared to DFS on only the programs which *both terminate* are given in fig. 4.1. As expected, BFS and IDFS use more memory in total than BDFS and DFS, while DFS is the fastest.

4.7. Effectiveness Analysis

4.7.1. Slicing

Type slice sizes (amount of highlighting) were calculated over the entire corpus. While *cast slice* sizes were calculated over the resulting elaborated expressions.

Figure 4.2 shows that type are generally small. In particular, the proportion of the context[3] highlighted is very low, generally less than 5% for dynamic code and 10% for annotated code. Therefore, they concisely explain the expressions' type on average.

Additionally, for errors, there are multiple inconsistent slices involved. Section 4.8.1 describes how these slices can be summarised to only report the inconsistent parts. We find that these *minimised* error slices are significantly (3x) smaller than directly *combining* the slices.

As hypothesised, combined slices within cast errors are smaller than combined static error slices on average (2x, fig. F.3). Therefore, casts can more precisely point to which part of an expressions type caused them.

4.7.2. Search Procedure

Witness Coverage

The search procedure terminates either with a witness or proving no witness exists. A majority of programs terminated when using BDFS, DFS and IDFS, with BDFS meeting the 75% target directly (fig. 4.3). IDFS and BFS perform relatively poorly likely due to excessive memory usage (fig. F.2).

However, not all static errors have a dynamic witness, e.g. errors within dead code. I manually classified each failed program for BDFS to check if a witness does exist, but was not found, or no witness exists. This gives only 2% of cases where BDFS failed to actually find an *existing* witness; DFS and IDFS also meet the goal of failing in less than 25% of cases. Section 4.8.3 goes into further detail on categorising the programs which time out, and how this could be avoided.

[3] Calculated by close approximation by the *program size*. As each program in the corpus is just one definition. Calculating the context itself is non-trivial.

Averages		Subdivisions						
			Combined Error Slices			Minimised Error Slices		
	unit	ok	expects	branches	all	expects	branches	all
Type Slice	size	8.2	13	22	15	5.7	3.2	5
Std. dev.		11	10	24	15	4.31	4.1	4.4
Proportion	%	5	8	14	9	3	2	3
Std. dev.		7	8	14	10	3	3	3
(Unannotated)								
Type Slice	size	7.5	21	133*	22.6	8.2	2.0*	8.2
Std. dev.		13	22	42*	25.2	12.6	0.0*	12.6
Proportion	%	4	14	0.48*	15	6	1*	5
Std. dev.		9	18	0.07*	18	9	0*	12
(Annotated)								

* only 2 annotated programs had inconsistent branches

Figure 4.2: Effectiveness: Type Slices

(a) DFS **(b)** Bounded DFS

(c) Interleaved DFS **(d)** BFS

Figure 4.3: Search Procedure Coverage

Witness & Trace Size

As predicted by the small-scope hypothesis, most programs admitted *small* witnesses (avg. sizes of 1-2, i.e. mostly base cases).

However, there was no linear correlation (Pearson correlation coefficient $= 0$) between witness sizes and trace sizes, even when normalised by the original deterministic evaluation trace lengths. This is likely because most errors are in the base cases, so few large witnesses are even found, with the noise from trace lengths to different programs' base cases dominating.

4.8. Critical Analysis

This section discusses the implications of the previous results and delves deeper into the reasoning behind them. As a response to this analysis, many improvements were devised.

(a) Ad-hoc Slice: Let expression omitted **(b)** Ordinary Slice

Figure 4.4: An Ad-hoc Slice vs. Ordinary Slice

4.8.1. Slicing

Type slicing theory (section 3.1) requires highlighted code to form valid expressions or contexts, though some highlighted parts, like unused or dynamic bindings, do not affect types and can be omitted. This motivated the use of unstructured (ad-hoc) slices (section 3.3.2).

In fig. 4.4, a bound integer `x` is only used in one branch. Since the other branch can already determine the type of the conditional, the `x` and `let` are excluded from the slice. Though the whole program is selected (see the red cursor) the let expression is omitted. A contribution slice would include everything, making it even more verbose.

Error Slices

Type errors arise from inconsistencies between a term's analysis and synthesis slices, or across synthesising branches. Understanding the error requires comparing all the slices involved.

Some type parts may agree, hence another form of type joining was introduced to isolate only the inconsistent parts. For instance, in fig. 4.5, differences like `(Int, Int)` vs `(Int, String)` highlight only the mismatched sub-slices.

(a) Partially Inconsistent Branches **(b)** Partially Inconsistent Expectations

Figure 4.5: Error Slices

Compound type inconsistencies necessarily differ at the outermost constructor (e.g., `List` vs `Int`), being the *primary* cause of the error. Deeper inconsistencies are not those causing the type error. Therefore, we could extract only the slice on the outermost constructor. These minimised slices (fig. 4.6) are significantly (avg. 3x) smaller than full combined slices (fig. 4.2). This ratio may grow with program complexity, the corpus had few partially inconsistent cases.

(a) Minimised Error Slice: Inner `Int` slice within list is omitted **(b)** Ordinary Error Slice

Figure 4.6: Minimised Error Slices

4.8.2. Static-Dynamic Error Correspondence

A static type error will place a term inside a cast error during elaboration, which can be associated with a dynamic error whose cast error is dependent on (decomposed from) this original failed cast. This works well for *inconsistent expectations* static type errors.

However, for inconsistent branches, no direct cast failure occurs until a branch result is used in a static context. Still, since elaboration adds casts to each branch, these can be tracked with the error. But, we cannot distinguish if the error was caused by the branch or the use.

4.8.3. Categorising Programs Lacking Type Error Witnesses

47 programs which timed out under the BDFS search procedure were manually inspected and classified as either:

- Witness Exists: BDFS failed to find an existing witness.
- Dead Code: The error lies in unreachable code:
  - Pattern Cast Failure: Error within a pattern matching branch, making the branch unreachable. These are detectable by the extended pattern directed instantiation algorithm (section 3.5.4).[4]
  - Unbound Constructor: Attemping to match an unbound constructor. Also detectable with the extended instantiation.
  - Wildcards: Erroneous code bound to the inaccessible wildcard pattern:
    `let _ = ... in ....`
  - Non-Trivial: Less easily detectable. One example exhibited this, infinitely recursing for all inputs.
- Hazel Bugs: Unboxing bugs present in the main branch (excluded from the statistics).

Figure 4.7 shows this distribution and three (paraphrased) examples are given in the appendix fig. F.5. The full classification is in `failure-classification.txt`.

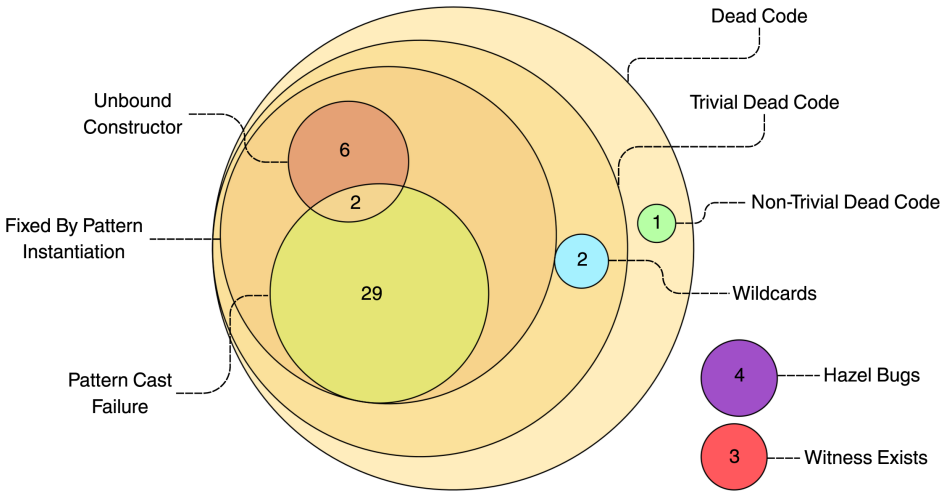


Figure 4.7: Distribution of Failed Program Classes

Non-Termination, Unfairness, and Search Order

DFS is fast but prone to non-termination, giving it lesser coverage than BDFS (fig. 4.3). If evaluation infinitely loops, so will DFS without exploring other instantiations. It is also unfair, it may never try some instantiations [54]. This affected 10% of cases.

To address this, BDFS, BFS, and IDFS were implemented, with BDFS performing the due to prioritising evaluation over instantiation. BFS and IDFS would do significantly better if they tried longer trace length more frequently, on average reaching only half the trace length (full data: fig. F.4). Wrapping evaluation in the evaluation algorithm would help this, but choosing how frequently to wrap evaluation is a delicate balancing act.

[4]The inconsistent patterns would be attempted, and subsequently reduced to expression cast failures.

Dead Code & Nested Errors

Errors within dead code cannot have a witness as they are not dynamically reachable. A significant portion (39/47) of failed programs for BDFS had trivial dead code which could easily be detected and returned explicitly as proving no witness exists.

Additionally, code can become dead due to errors. 12 (32%) of the dead code classified had a nested error within a branch that is unreachable due to a pattern error on the branch pattern. Even if witnesses are found for these branch errors, the nested errors remain hidden.

Cast Laziness

Hazel treats casts lazily, deferring cast transitions on compound data until their elements are used (e.g., tuple elements). Hazel does not detect cast errors between these non-ground compound types (e.g., `[Int]` vs `[String]`) until elements are used. Therefore if such elements are unused in code, the error cannot be witnessed. These are uncommon, none appearing in the search corpus.

Addressing this would require eager cast semantics, as has been previously explored for dynamic and gradual type systems [67, 46] (often referred to as coercions). Additionally, they would catch dynamic errors earlier, for example, fig. 4.8 shows a cast inconsistency undetected (by lazy casts) at runtime until the tuple element is accessed.

```
let x = (1, "str") in
let f = (fun x : (Int, Int) -> x)
in f(x)

≡ (1, "str"): (○, ○): (Int, Int)
```

Figure 4.8: Inconsistent Lazy Casts: No Cast Failure

Combinatorial Explosion

When multiple holes are involved in searching, the search space increases exponentially. This combinatorial explosion especially impacts IDFS and BFS, who prioritise instantiation over evaluation. This leads to high memory use and hinders evaluation from progressing far enough to detect errors, even when valid witnesses have been instantiated.

Further, some errors only arise on very specific inputs, e.g. `(23, 31)` and `(31, 23)` in fig. 4.9. Directing instantiation to maximise code coverage earlier could find such errors attempting fewer instantiations.

4.8.4. Improving Code Coverage

The search procedure struggles when compound witnesses require specific, interdependent parts, being much less likely to instantiate. These errors are also harder for programmers to detect or understand; understanding the error might require recognizing input interdependencies.

Intelligently directing hole instantiations to better cover the code would help. A pattern-directed instantiation as described in section 3.5.4 would discover 2 of the 3 missed witnesses by BDFS.

Still, BDFS retains a very high coverage over the search procedure when excluding dead code (97%).

```
let f = fun (x, y) ->
  if x * y == 713 then 1 + "str" else 0
in ○
```

Figure 4.9: Witness requiring very specific instantiations

4.9. Holistic Evaluation

This section considers a number of examples of ill-typed Hazel programs, *holistically* and *qualitatively* evaluating how a user might use the three features and the existing bidirectional type error localisation [16] to debug the errors.

4.9.1. Interaction with Existing Hazel Type Error Localisation

Hazel has three error types addressed by this project: *inconsistent expectations* (analytic and synthetic types are inconsistent), *inconsistent branches* (branches or list element types are inconsistent), and *inexhaustive matches*.

Section 4.8.2 showed how witnesses can be associated to inconsistency errors. While indeterminate evaluation can give examples for pattern inexhaustivity, standard pattern matrix methods [50] are more efficient. The matrices would also be useful for directing pattern instantiation.

When errors arise from the programmer *misunderstanding* the program types, error localisation can be inaccurate (due to assuming different types to the programmer's expectations). The context inspector (fig. 4.10) clarifies what assumptions the system makes while slicing and witnesses can explain *why*.

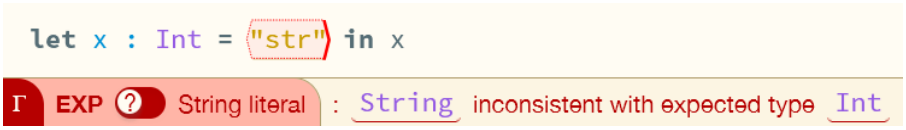
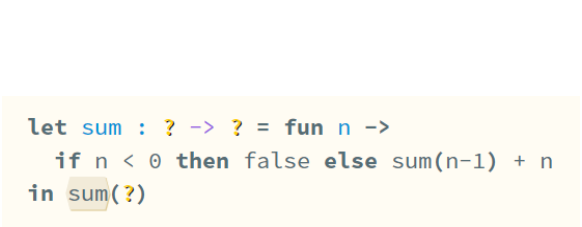


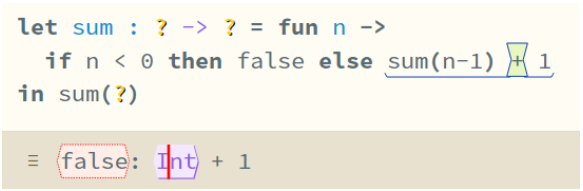
Figure 4.10: Selected Static Error described by Hazel Context Inspector

When the programmer and system agree on the types, bidirectional typing generally localises the error(s) well [36, 16]. However, there is not always enough static information to even recognise errors, and many type annotations may need to be inserted to detect the error. The search procedure tests such code for such type errors automatically, and can be a quicker way to detect these errors than adding annotations.

4.9.2. Examples

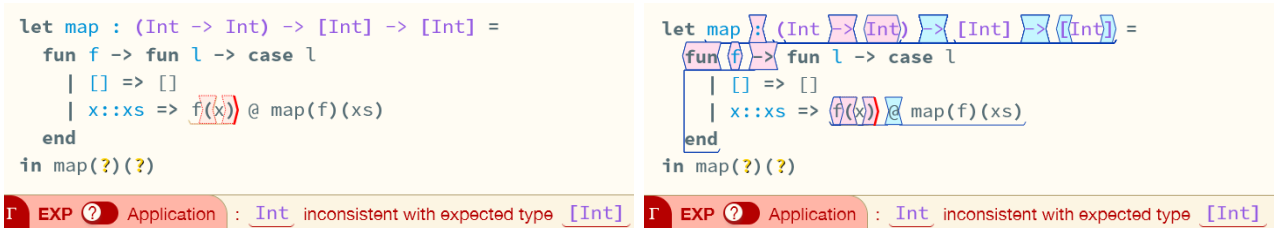


(a) Error



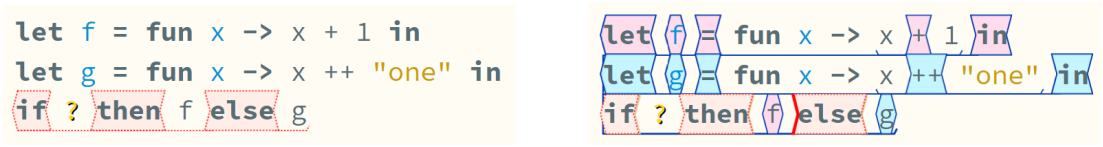
(b) Automated witness found with cast slice blaming + operator

Figure 4.11: Dynamic Code: An error is not statically found. Cast slicing still works even without the type annotations, blaming addition forcing ints.



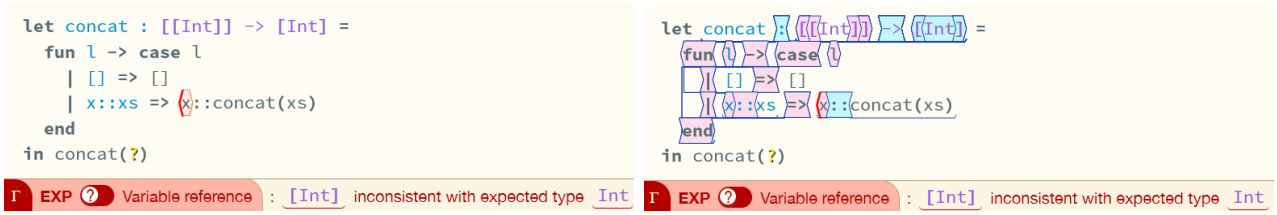
(a) Error **(b)** With Error Slice

Figure 4.12: Inconsistent Expectations: `f(x)` synthesises type `Int` but analyses against `[Int]`. The *minimised* error slice shows (in pink) why `Int` is synthesised (due to input `f` being annotated `Int -> Int` and applied) and (in blue) why a list is expected.



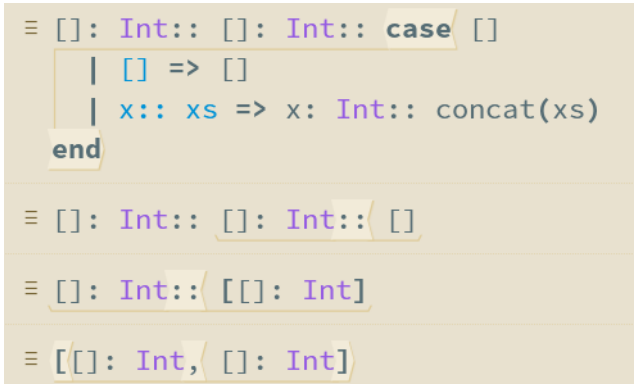
(a) Error **(b)** With Error Slice

Figure 4.13: Inconsistent Branches: A more complex inconsistency involving non-local bindings. *Minimal* error slices highlight only the bindings, and conflicting addition and string concatenation operators.

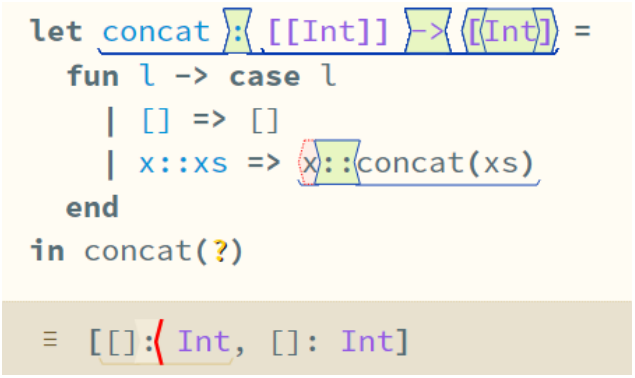


(a) The user thinks that list cons `::` concatenates lists.

(b) Error slice highlights the offending `::`, but with considerable noise distracting from this. Also, if the user still expects `::` to perform concatenation, this is useless.



(c) The user cycles through increasingly larger witnesses (until `[[], []]`) to spot the consing pattern. End of trace concretely shows `::` acting as cons.



(d) Alternatively, the cast slice to `Int` concisely retrieves the relevant part of the original type slice.

Figure 4.14: Holistic Example involving all three features

```
let fold : (Int -> Int -> Int) -> Int -> [Int] -> Int
= fun f -> fun init -> fun l ->
  case l
    | [] => init
    | x::xs => f(x)(fold(f)(init)(xs))
  end
in
let add = fun (x : Int, y : Int) -> x + y in
let sum = fold(add)(0)
in sum(?)
```

(a) Fold takes curried functions, but uncurried add is used.

```
let fold : (Int -> Int -> Int) -> Int -> [Int] -> Int
= fun f -> fun init -> fun l ->
  case l
    | [] => init
    | x::xs => f(x)(fold(f)(init)(xs))
  end
in
let add = fun (x : Int, y : Int) -> x + y in
let sum = fold(add)(0) in
sum(?)

≡ (<add>)(0: ((  ,  )): ○ -> ○)(0)
```

(c) Witness (`[0]`) found automatically: expects input 0 into `add` to be a tuple. Cast slice considers only the `(Int, Int)` $\not\sim$ `Int` inconsistency.

```
let fold : (Int -> Int -> Int) -> Int -> [Int] -> Int
= fun f -> fun init -> fun l ->
  case l
    | [] => init
    | x::xs => f(x)(fold(f)(init)(xs))
  end
in
let add = fun (x : Int, y : Int) -> x + y in
let sum = fold(add)(0) in
sum(?)
```

(b) Error slice is verbose, and merges multiple inconsistencies: `(Int, Int)` $\not\sim$ `Int` and `Int` $\not\sim$ `Int` $\to$ `Int`. The second inconsistency is minimised to `Int` $\not\sim$ `?` $\to$ `?`.

```
let fold : (Int -> Int -> Int) -> Int -> [Int] -> Int
= fun f -> fun init -> fun l ->
  case l
    | [] => init
    | x::xs => f(x)(fold(f)(init)(xs))
  end
in
let add = fun (x : Int, y : Int) -> x + y in
let sum = fold(add)(0) in
sum(?)

≡ (<add>)(0: (○, ○)): ○ -> ○)(0)
```

(d) Same witness has a second cast error: output of `add(0)` is expected to be a function. Cast slice considers only the minimised `Int` $\not\sim$ `?` $\to$ `?` inconsistency.

Figure 4.15: A more subtle holistic example involving currying. Requires slice decomposition internally.

Conclusions

All core goals and extensions set out in section 2.3 were achieved except for the low priority extensions, of which extended pattern instantiation was partially implemented. All must/should/could have features of Hazel were supported except for only partial support for labelled tuples and type substitution.

Type slicing proved more expressive than initially planned, applying to *all* expressions, *not just errors*. Therefore, it can also be used to learn and build understanding on how Hazel's type system works, not just for debugging errors.

An indeterminate evaluation framework (section 3.5) was built, greatly generalising the original search procedure [32] to allow multiple search orderings, and different classes of results, e.g. concrete values, or derived results like expression sizes.

5.1. Further Directions

5.1.1. UI Improvements, User Studies

During the holistic evaluation, various usability and UI improvements were considered and added as low-priority extensions (appendix G). Once implemented, a user study could assess the real world effectiveness of type slicing and cast slicing in improving understanding of the type system, and locating errors.

5.1.2. Cast Slicing

Cast slicing propagates type slice information throughout evaluation. But, it provide no slicing on *execution properties* of the program. Extended slicing methods could, for example, provide a minimal program which evaluates to the *same* cast around the *same* value. This could build on traditional *dynamic slicing* methods [74, 43], and find use also in debugging *semantic* errors.

5.1.3. Property Testing

Indeterminate evaluation provides a way to generate inputs to function. So, it could be used for property testing, generating inputs to functions and testing expressions. A framework similar to SmallCheck [49] or QuickCheck [60] could be implemented. Being part of the evaluator, intermediate expressions and execution properties could also be tested.

5.1.4. Non-determinism, Connections to Logic Programming

Non-deterministic evaluation could be harnessed to implement non-deterministic constructs (e.g. a choice operator) for Hazel. Treating holes as unknowns and instantiating lazily is reminiscent of *free logic variables* in functional logic programming languages [45], e.g. in *Curry* [9]. Adding unification [72] would allow full logic programming in Hazel. A needed-narrowing evaluation strategy [59] would be a significantly more efficient than the current lazy non-deterministic instantiation.

5.1.5. Symbolic Execution

Use of symbolic Execution [25] would improve the proportion of witnesses found by the search procedure by improving code coverage (section 4.8.4). Indeterminate evaluation can be considered a form of simple symbolic execution, only considering constraints enforced by casts. Section 3.5.4 considered directing instantiation using patterns within match statements.

5.1.6. Let Polymorphism & Global Inference

Types in globally inferred languages are often more subtle, as there are fewer annotations asserting the programmers intent. This project would be particularly useful in these situations.

In Hazel: Global inference is difficult or impossible to combine with complex type systems. Global inference for high rank polymorphism, like Hazel's System-F style polymorphism, is undecidable [64]. However, a form of let-polymorphism via principal type schemes [77] is possible. Intersecting this with gradual typing has been explored by Garcia and Cimini [33]

and Miyazaki et al. [23], the latter of which would integrate most smoothly with Hazel and indeterminate evaluation.

Constraint Slicing: The search procedure and cast slicing are relatively easily extended to a let-polymorphic Hazel in the style of Miyazaki et al. However, type slicing now has to consider non-local constraints alongside code which sourced them, differing significantly from bidirectional type slicing. Somewhat similar ideas, but limited only to errors, have been explored in *constraint-based type error slicing* by Haack and Wells [55].

5.2. Lessons Learnt

This project required extensive theory and mathematical thought, requiring in extensive reading of the surrounding research. I have, therefore, learnt how to effectively find *relevant* research and quickly extract and understand their key contributions.

Further, it worked on the rapidly moving open-source codebase Hazel. As such, I have learnt how a large software project works, learning how to use effectively use version control and continuous integration tools. Additionally, communicating with other developers has offered insights into how collaborative development of software works.

The result was an extensive multi-faceted project, requiring great effort. On reflection, it might have been better to focus on either type slicing *or* the search procedure, allowing time to turn one of these into a truly *usable* debugging aid (implementing the UI extensions). In retrospect, it would have been more time-efficient and rigorous to fully automate slicing regression tests, creating a framework that abstracts away the randomised term IDs within slices.

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Overview of Semantics and Type Systems

Syntax

Expression-based language syntax are the core foundation behind formal reasoning of programming language semantics and type systems. Typically languages are split into expressions e, constants c, variables x, and types τ. Hazel additionally defines patterns p.

A typical lambda calculus can recursively define it's grammar in the following form:

$$b ::= \text{A set of base types}$$

$$\tau ::= \tau \to \tau \mid b$$

$$c ::= \text{A set of constants of base types}$$

$$x ::= \text{A set of variable names}$$

$$e ::= c \mid x \mid \lambda x : \tau . e \mid e(e)$$

Judgements & Inference Rules

A *judgement*, J, is an assertion about *expressions* in a language [30]. For example:

- Exp e – e is an *expression*
- n : `int` – n has type `int`
- $e \Downarrow v$ – e evaluates to *value* v

While an *inference rule* is a collection of judgements $J, J_1, \ldots, J_n$:

$$\frac{J_1 \quad J_2 \quad \ldots \quad J_n}{J}$$

Representing the *rule* that if the *premises*, $J_1, \ldots, J_n$ are true then the conclusion, J, is true. When the collection of premises is empty, it is an *axiom* stating that the judgement is *always* true. Truth of a judgement J can be assessed by constructing a *derivation*, a tree of rules where it's leaves are axioms. It is then possible to use rules to define a judgement by taking the largest judgement that is *closed* under a collection of rules. This gives the result that a judgement J is true *if and only if* it has a derivation.

Properties on expressions can be proved using *rule induction*, if a property is *preserved* by every rule for a judgement, and true for it's axioms, then the property holds whenever the judgement is derivable.

A *hypothetical judgement* is a judgement written as:

$$J_1, \ldots, J_n \vdash J$$

is true if J is derivable when additionally assuming each J_i are axioms. Often written $\Gamma \vdash J$ and read J *holds under context* Γ. Hypothetical judgements can be similarly defined inductively via *rules*.

Defining a Type System

A typical type system can be expressed by defining the following hypothetical judgement form $\Gamma \vdash e : \tau$ read as *the expression* e *has type* τ *under typing context* Γ and referred as a *typing judgement*. Here, $e : \tau$ means that expression e has type τ. The *typing assumptions*, Γ, is a *partial function*[1] from variables to types for variables, notated $x_1 : \tau_1, \ldots, x_n : \tau_2$. For example the SLTC[2] [57, ch. 9] has a typing rule for lambda expression and application as follows:

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x.\ e : \tau_1 \to \tau_2} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1(e_2) : \tau_2}$$

[1] A function, which may be *undefined* for some inputs, notated $f(x) = \bot$.

[2] Simply typed lambda calculus.

Meaning, $\lambda x.e$ has type $\tau_1 \rightarrow \tau_2$ if e has type τ_2 under the extended context additionally assuming that x has type τ_1. And, $e_1(e_2)$ has type τ_2 if e_1 is a function of type $\tau_1 \rightarrow \tau_2$ and its argument e_2 has type τ_1.

Small Step Operational Semantics

A relation $e_1 \rightarrow e_2$ can be defined via judgement rules to determine the evaluation semantics of the language. It's multi-step (transitive closure) analogue is notated $e_1 \rightarrow^* e_2$, meaning $e_1 = e_2$ or there exists a sequence of steps $e_1 \rightarrow e_1' \rightarrow ... \rightarrow e_2$.

$$\frac{}{e \rightarrow^* e} \qquad \frac{e_1 \rightarrow e_2 \quad e_2 \rightarrow^* e_3}{e_1 \rightarrow^* e_3}$$

Evaluation order can be controlled by considering classifying terms into *normal forms* (values). A call by value language would consider normal forms v as either constants or functions:

$$v ::= c \mid \lambda x : \tau.e$$

Hazel evaluations around holes by treating them as normal forms (final forms). A call by value semantics for the lambda calculus would include, where $[v/x]e$ is capture avoiding substitution of value v for variable x in expression e:

$$\frac{}{(\lambda x : \tau.\ e)v \rightarrow [v/x]e} \qquad \frac{e_2 \rightarrow e_2'}{e_1(e_2) \rightarrow e_1(e_2')}$$

Hazel Formal Semantics

This is the complete formal semantics for the Hazel *core calculus*. It is *gradually typed* so consists of both an *external language* and *internal language*.

B.1. Syntax

$$\tau ::= b \mid \tau \to \tau \mid \ ?$$
$$e ::= c \mid x \mid \lambda x : \tau.e \mid \lambda x.e \mid e(e) \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid e : \tau$$
$$d ::= c \mid x \mid \lambda x : \tau d \mid d(d) \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow ? \not\Rightarrow \tau \rangle$$

Figure B.1: Syntax: *types* τ, *external expressions* e, *internal expressions* d. With x ranging over variables, u over hole names, σ over $x \to d$ *internal language* substitutions/environments, b over base types and c over constants.

B.2. Static Type System

B.2.1. External Language

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesises type τ under context Γ

$$\text{SConst}\ \frac{}{\Gamma \vdash c \Rightarrow b} \qquad \text{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \text{SFun}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ e \Rightarrow \tau_1 \to \tau_2}$$

$$\text{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \text{SEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow ?}$$

$$\text{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow ?} \qquad \text{SAsc}\ \frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyses against type τ under context Γ

$$\text{AFun}\ \frac{\tau \blacktriangleright_\to \tau_1 \to \tau_2 \qquad \Gamma, x : \tau_1 \vdash e \Leftarrow \tau_2}{\Gamma \vdash \lambda x.e \Leftarrow \tau} \qquad \text{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau'}$$

Figure B.2: Bidirectional typing judgements for *external expressions*

$\boxed{\tau_1 \sim \tau_2}$ τ_1 is consistent with τ_2

$$\text{TCDyn1}\ \frac{}{? \sim \tau} \qquad \text{TCDyn2}\ \frac{}{\tau \sim ?} \qquad \text{TCRfl}\ \frac{}{\tau \sim \tau} \qquad \text{TCFun}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

Figure B.3: Type consistency

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has arrow type $\tau_1 \to \tau_2$

$$\text{MADyn}\ \frac{}{? \blacktriangleright_{\to} ? \to ?} \qquad \text{MAFun}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

Figure B.4: Type Matching

B.2.2. Elaboration

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e syntheses type τ and elaborates to d

$$\text{ESConst}\ \frac{}{\Gamma \vdash c \Rightarrow b \rightsquigarrow c \dashv \emptyset} \qquad \text{ESVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\text{ESFun}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : \tau_1.e \Rightarrow \tau_1 \to \tau_2 \rightsquigarrow \lambda x : \tau_1.d \dashv \Delta}$$

$$\text{ESApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_2 \to \tau \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\text{ESEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow ? \rightsquigarrow (\!|\,|\!)^u_{\mathrm{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}$$

$$\text{ESNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Rightarrow ? \rightsquigarrow (\!|d|\!)^u_{\mathrm{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}$$

$$\text{ESAsc}\ \frac{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau \rangle \dashv \Delta}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$ e analyses against type τ and elaborates to d of consistent type τ'

$$\text{EAFun}\ \frac{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2 \qquad \Gamma, x : \tau_1 \vdash e \Leftarrow \tau_2 \rightsquigarrow d : \tau_2' \dashv \Delta}{\Gamma \vdash \lambda x.e \Leftarrow \tau \rightsquigarrow \lambda x : \tau_1.d : \tau_1 \to \tau_2' \dashv \Delta}$$

$$\text{EASubsume}\ \frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

$$\text{EAEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{\mathrm{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\text{EANEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathrm{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

Figure B.5: Elaboration judgements

$$id(x_1 : \tau_1, \ldots, x_n : \tau_n) := [x_1/x_1, \ldots, x_n/x_n]$$

$\Delta; \Gamma \vdash \sigma : \Gamma'$ iff $\mathrm{dom}(\sigma) = \mathrm{dom}(\Gamma')$ and for every $x : \tau \in \Gamma'$ then: $\Delta; \Gamma \vdash \sigma(x) : \tau$

Figure B.7: Identity substitution and substitution typing

B.2.3. Internal Language

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\textsc{TACons}\ \frac{}{\Delta; \Gamma \vdash c : b} \qquad \textsc{TAVar}\ \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \qquad \textsc{TAFun}\ \frac{\Delta; \Gamma, x : \tau_1 \vdash d : \tau_2}{\Delta; \Gamma \vdash \lambda x : \tau_1 . d : \tau_1 \to \tau_2}$$

$$\textsc{TAApp}\ \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \quad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau} \qquad \textsc{TAEHole}\ \frac{u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\,|\!)^u_\sigma : \tau}$$

$$\textsc{TANEHole}\ \frac{\Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^u_\sigma : \tau} \qquad \textsc{TACast}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

$$\textsc{TACastError}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow ? \nRightarrow \tau_2 \rangle : \tau_2}$$

Figure B.6: Type assignment judgement for *internal expressions*

$\boxed{\tau \text{ ground}}$ τ is a ground type

$$\textsc{GBase}\ \frac{}{b \text{ ground}} \qquad \textsc{GDynFun}\ \frac{}{? \to ? \text{ ground}}$$

Figure B.8: Ground types

B.3. Dynamics

B.3.1. Final Forms

$\boxed{d \text{ final}}$ d is final

$$\text{FBoxedVal}\frac{d \text{ boxedval}}{d \text{ final}} \qquad \text{FIndex}\frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\text{VConst}\frac{}{c \text{ val}} \qquad \text{VFun}\frac{}{\lambda x : \tau.d \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\text{BVVal}\frac{d \text{ val}}{d \text{ boxedval}} \qquad \text{BVFunCast}\frac{\tau \to \tau_2 \neq \tau_3 \to \tau_4 \quad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

$$\text{BVDynCast}\frac{d \text{ boxedval} \quad \tau \text{ ground}}{d\langle \tau \Rightarrow ? \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\text{IEHole}\frac{}{(\!|\,|\!)^u_\sigma \text{ indet}} \qquad \text{INEHole}\frac{d \text{ final}}{(\!|d|\!)^u_\sigma \text{ indet}} \qquad \text{IAp}\frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \quad d_1 \text{ indet} \quad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

$$\text{ICastGD}\frac{d \text{ indet} \quad \tau \text{ ground}}{d\langle \tau \Rightarrow ? \rangle \text{ indet}} \qquad \text{ICastDG}\frac{d \neq d'\langle \tau' \Rightarrow ? \rangle \quad d \text{ indet} \quad \tau \text{ ground}}{d\langle ? \Rightarrow \tau \rangle \text{ indet}}$$

$$\text{ICastFun}\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \quad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}} \qquad \text{ICastError}\frac{d \text{ final} \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow ? \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

Figure B.9: Final forms

B.3.2. Instructions

$\boxed{d \longrightarrow d'}$ d takes and instruction transition to d'

$$\text{ITFun}\frac{}{(\lambda x : \tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \text{ITCastId}\frac{}{d\langle \tau \Rightarrow \tau \rangle \longrightarrow d}$$

$$\text{ITAppCast}\frac{\tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle(d) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}$$

$$\text{ITCast}\frac{\tau \text{ ground}}{d\langle \tau \Rightarrow ? \Rightarrow \tau \rangle \longrightarrow d} \qquad \text{ITCastError}\frac{\tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d\langle \tau_1 \Rightarrow ? \Rightarrow \tau_2 \rangle \longrightarrow d\langle \tau_1 \Rightarrow ? \not\Rightarrow \tau_2 \rangle}$$

$$\text{ITGround}\frac{\tau \blacktriangleright_{\text{ground}} \tau'}{d\langle \tau \Rightarrow ? \rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow ? \rangle} \qquad \text{ITExpand}\frac{\tau \blacktriangleright_{\text{ground}} \tau'}{d\langle ? \Rightarrow \tau \rangle \longrightarrow d\langle ? \Rightarrow \tau' \Rightarrow \tau \rangle}$$

Figure B.10: Instruction transitions (non-deterministic version)

B.3.3. Contextual Dynamics

$\boxed{\tau \blacktriangleright_{\text{ground}} \tau'}$ τ matches ground type τ'

$$\frac{\tau_1 \to \tau_2 \neq ? \to ?}{\tau_1 \to \tau_2 \blacktriangleright_{\text{ground}} ? \to ?}$$

Figure B.11: Ground type matching

Context syntax:

$$E ::= \circ \mid E(d) \mid d(E) \mid (\!|E|\!)^u_\sigma \mid E\langle \tau \Rightarrow \tau \rangle \mid E\langle \tau \Rightarrow ? \not\Rightarrow \tau \rangle$$

$\boxed{d = E[d]}$ d is the context E filled with d' in place of $\circ$

$$\text{ECOuter}\frac{}{d = \circ[d]} \qquad \text{ECApp1}\frac{d_1 = E[d_1']}{d_1(d_2) = E(d_2)[d_1]} \qquad \text{ECApp2}\frac{d_2 = E[d_2]}{d_1(d_2) = d_1(E)[d_2']}$$

$$\text{ECNEHole}\frac{d = E[d']}{(\!|d|\!)^u_\sigma = (\!|E|\!)^u_\sigma[d']} \qquad \text{ECCast}\frac{d = E[d']}{d\langle \tau_1 \Rightarrow \tau_2 \rangle = E\langle \tau_1 \Rightarrow \tau_2 \rangle[d']}$$

$$\text{ECCastError}\frac{d = E[d']}{d\langle \tau_1 \Rightarrow ? \not\Rightarrow \tau_2 \rangle = E\langle \tau_1 \Rightarrow ? \not\Rightarrow \tau_2 \rangle[d']}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\text{Step}\frac{d_1 = E[d_2] \qquad d_2 \longrightarrow d_2' \qquad d_1' = E[d_2']}{d_1 \mapsto d_1'}$$

Figure B.12: Contextual dynamics of the internal language (non-determinism version)

B.3.4. Hole Substitution

$\boxed{[\![d/u]\!]d' = d''}$ d'' is d' with each hole u substituted with d in the respective hole's environment σ.

$$
\begin{array}{lll}
[\![d/u]\!]c & = c & \\
[\![d/u]\!]x & = x & \\
[\![d/u]\!]\lambda x : \tau.d' & = \lambda x : \tau.[\![d/u]\!]d' & \\
[\![d/u]\!]d_1(d_2) & = ([\![d/u]\!]d_1)([\![d/u]\!]d_2) & \\
[\![d/u]\!](\!|)^u_\sigma & = [[\![d/u]\!]\sigma]d & \\
[\![d/u]\!](\!|)^v_\sigma & = (\!|)^b_{[\![d/u]\!]\sigma} & \text{if } u \neq v \\
[\![d/u]\!](\!|d'|\!)^u_\sigma & = [[\![d/u]\!]\sigma]d & \\
[\![d/u]\!](\!|d'|\!)^v_\sigma & = (\!|[\![d/u]\!]d'|\!)^b_{[\![d/u]\!]\sigma} & \text{if } u \neq v \\
[\![d/u]\!]d'\langle \tau \Rightarrow \tau' \rangle & = ([\![d/u]\!]d')\langle \tau \Rightarrow \tau' \rangle & \\
[\![d/u]\!]d'\langle \tau \Rightarrow ? \not\Rightarrow \tau' \rangle & = ([\![d/u]\!]d')\langle \tau \Rightarrow ? \not\Rightarrow \tau' \rangle &
\end{array}
$$

$\boxed{[\![d/u]\!]\sigma = \sigma'}$ σ' is σ with each hole u in σ substituted with d in the respective hole's environment.

$$
\begin{aligned}
[\![d/u]\!]\cdot &= \cdot \\
[\![d/u]\!]\sigma, d'/x &= [\![d/u]\!]\sigma, ([\![d/u]\!]d')/x
\end{aligned}
$$

Figure B.13: Hole substitution

Slicing Theory

C.1. Expression Typing Slices

C.1.1. Term Slices

Syntax

Extending core Hazel syntax with patterns $pt = \_ \mid x$ where $\_$ is the wildcard pattern (binding the argument to nothing).

Definition 11 (Term Slice Syntax)**.** *Pattern expression slices p:*

$$p ::= \square_{pat} \mid \_ \mid x$$

Type slices υ:

$$\upsilon ::= \square_{typ} \mid ? \mid b \mid \upsilon \to \upsilon$$

Expression slices ς:

$$\varsigma ::= \square_{exp} \mid c \mid x \mid \lambda p : \upsilon.\ \varsigma \mid \lambda p.\ \varsigma \mid \varsigma(\varsigma) \mid (\!|\,|\!)^u \mid (\!|\varsigma|\!)^u \mid \varsigma : \upsilon$$

Note: labels for gaps are generally omitted, being determined from their position.

Precision Relation

Definition 12 (Term Precision)**.** *Pattern slices p:*

$$\frac{}{\square_{pat} \sqsubseteq p} \qquad \frac{}{x \sqsubseteq x}$$

Type slices υ:

$$\frac{}{\square_{typ} \sqsubseteq \upsilon} \qquad \frac{}{\upsilon \sqsubseteq \upsilon} \qquad \frac{\upsilon_1' \sqsubseteq \upsilon_1 \quad \upsilon_2' \sqsubseteq \upsilon_2}{\upsilon_1' \to \upsilon_2' \sqsubseteq \upsilon_1 \to \upsilon_2}$$

Expression slices ς:

$$\frac{}{\square_{exp} \sqsubseteq \varsigma} \qquad \frac{}{\varsigma \sqsubseteq \varsigma} \qquad \frac{p' \sqsubseteq p \quad \upsilon' \sqsubseteq \upsilon \quad \varsigma' \sqsubseteq \varsigma}{\lambda p' : \upsilon'.\ \varsigma' \sqsubseteq \lambda p : \upsilon.\ \varsigma}$$

$$\frac{p' \sqsubseteq p \quad \varsigma' \sqsubseteq \varsigma}{\lambda p'.\ \varsigma' \sqsubseteq \lambda p.\ \varsigma} \qquad \frac{\varsigma_1' \sqsubseteq \varsigma_1 \quad \varsigma_2' \sqsubseteq \varsigma_2}{\varsigma_1'(\varsigma_2') \sqsubseteq \varsigma_1(\varsigma_2)} \qquad \frac{\varsigma' \sqsubseteq \varsigma \quad \upsilon' \sqsubseteq \upsilon}{\varsigma' : \upsilon' \sqsubseteq \varsigma : \upsilon}$$

Proposition 5 (Precision is a partial order)**.** *Term precision forms a partial order on term slices. That is:*

$$\frac{}{\varsigma \sqsubseteq \varsigma} \qquad \frac{\varsigma_1 \sqsubseteq \varsigma_2 \quad \varsigma_2 \sqsubseteq \varsigma_1}{\varsigma_1 = \varsigma_2} \qquad \frac{\varsigma_1 \sqsubseteq \varsigma_2 \quad \varsigma_2 \sqsubseteq \varsigma_3}{\varsigma_1 \sqsubseteq \varsigma_3}$$

Proof. Trivial rule induction on the structure of $\sqsubseteq$. □

Lattice Structure

Definition 13 (Term Slice Joins)**.** *For all t:*

$$\square \sqcup t = t = t \sqcup \square \qquad t \sqcup t = t$$

$$(\upsilon_1 \to \upsilon_2) \sqcup (\upsilon_1' \to \upsilon_2') = (\upsilon_1 \sqcup \upsilon_1') \to (\upsilon_2 \sqcup \upsilon_2')$$

$$(\lambda p : \upsilon.\ \varsigma) \sqcup (\lambda p' : \upsilon'.\ \varsigma') = \lambda(p \sqcup p') : (\upsilon \sqcup \upsilon').\ (\varsigma \sqcup \varsigma') \qquad (\lambda p.\ \varsigma) \sqcup (\lambda p'.\ \varsigma') = \lambda(p \sqcup p').\ (\varsigma \sqcup \varsigma')$$

$$\varsigma_1(\varsigma_2) \sqcup \varsigma_1'(\varsigma_2') = (\varsigma_1 \sqcup \varsigma_1')(\varsigma_2 \sqcup \varsigma_2') \qquad (\varsigma : \upsilon) \sqcup (\varsigma' : \upsilon') = (\varsigma \sqcup \varsigma') : (\upsilon \sqcup \upsilon')$$

Definition 14 (Term Slice Meets)**.** *For all t, t':*

$$\square \sqcap t = \square = t \sqcap \square \qquad t \sqcap t = t$$

$$(\upsilon_1 \to \upsilon_2) \sqcap (\upsilon_1' \to \upsilon_2') = (\upsilon_1 \sqcap \upsilon_1') \to (\upsilon_2 \sqcap \upsilon_2')$$

$$(\lambda p : \upsilon.\ \varsigma) \sqcap (\lambda p' : \upsilon'.\ \varsigma') = \lambda(p \sqcap p') : (\upsilon \sqcap \upsilon').\ (\varsigma \sqcap \varsigma') \qquad (\lambda p.\ \varsigma) \sqcap (\lambda p'.\ \varsigma') = \lambda(p \sqcap p').\ (\varsigma \sqcap \varsigma')$$

$$\varsigma_1(\varsigma_2) \sqcap \varsigma_1'(\varsigma_2') = (\varsigma_1 \sqcap \varsigma_1')(\varsigma_2 \sqcap \varsigma_2') \qquad (\varsigma : \upsilon) \sqcap (\varsigma' : \upsilon') = (\varsigma \sqcap \varsigma') : (\upsilon \sqcap \upsilon')$$

Otherwise:

$$t \sqcap t' = \square$$

Proposition 6 (Term Slice Bounded Lattice)**.** *For any term t, the set of slices ς_1 and ς_2 of t forms a bounded lattice:*

- *The join, $\varsigma_1 \sqcup \varsigma_2$ exists , being an upper bound for ς_1, ς_2:*

  $$\varsigma_1 \sqsubseteq \varsigma_1 \sqcup \varsigma_2 \qquad \varsigma_2 \sqsubseteq \varsigma_1 \sqcup \varsigma_2$$

  And, any other slice $\varsigma \sqsubseteq t$ which is also an upper bound of ς_1, ς_2 is more or equally precise than the join:

  $$\varsigma_1 \sqcup \varsigma_2 \sqsubseteq \varsigma$$

- *The meet, $\varsigma_1 \sqcap \varsigma_2$ exists, being a lower bound for ς_1, ς_2:*

  $$\varsigma_1 \sqcap \varsigma_2 \sqsubseteq \varsigma_1 \qquad \varsigma_1 \sqcap \varsigma_2 \sqsubseteq \varsigma_2$$

  And, any other slice $\varsigma \sqsubseteq t$ which is also a lower bound of ς_1, ς_2 is less or equally precise than the meet:

  $$\varsigma \sqsubseteq \varsigma_1 \sqcup \varsigma_2$$

- *And the join and meet operations satisfy the absorption laws for any slice ς:*

  $$\varsigma_1 \sqcap (\varsigma_1 \sqcup \varsigma_2) = \varsigma_1 \qquad \varsigma_1 \sqcup (\varsigma_1 \sqcap \varsigma_2) = \varsigma_1$$

  And idempotent laws:

  $$\varsigma_1 \sqcup \varsigma_1 = \varsigma_1 = \varsigma_1 \sqcap \varsigma_1$$

- *The lattice is bounded. That is, $\square \sqsubseteq \varsigma \sqsubseteq t$.*

Proof. Obvious but tedious. Requires proving the definitions above correctly make joins as least upper bounds and meets as greatest lower bounds. Existence follows from there always being a lower bound $\square$ and every slice being of a complete term t. □

C.1.2. Typing Assumption Slices

Typing Assumptions as Partial Functions

Definition 15 (Typing Assumptions)**.** *A typing assumption function Γ is a partial function from the set of variables $\mathcal{X}$ to types $\mathcal{T}$. A partial function is either defined $\Gamma(x) = \tau$ or undefined $\Gamma(x) = \bot$. It's domain $\mathrm{dom}(\Gamma)$ is largest the set of variables $S \subseteq \mathcal{X}$ for which Γ is defined: $\forall x \in S.\ \Gamma(x) \neq \bot$.*

Typing Assumption Slices

Definition 16 (Typing Assumption Slices)**.** *A typing assumption slice γ is a partial function from the set of variables $\mathcal{X}$ to type slices υ.*

Precision

Definition 17 (Typing Assumption Slice Precision)**.** *For typing assumption slices γ_1, γ_2. Where* $\mathrm{dom}(f)$ *is the set of variables for which a partial function f is defined:*

$$\gamma_1 \sqsubseteq \gamma_2 \iff \mathrm{dom}(\gamma_1) \subseteq \mathrm{dom}(\gamma_2) \text{ and } \forall x \in \mathrm{dom}(\gamma_1).\ \gamma_1(x) \sqsubseteq \gamma_2(x)$$

Proposition 7 (Precision is a partial order)**.** *Typing assumption precision forms a partial order on typing assumption slices.*

Proof. Follows from $\subseteq$ and $\sqsubseteq$ being partial orders, and that logical conjunction distributes over forall quantifiers. □

Lattice Structure

Definition 18 (Typing Assumption Slice Joins and Meets)**.** *For typing slices γ_1, γ_2, and any variable x:*

- *If $\gamma_1(x) = \bot$ then $(\gamma_1 \sqcup \gamma_2)(x) = \gamma_2(x)$ and $(\gamma_1 \sqcap \gamma_2)(x) = \bot$.*
- *If $\gamma_2(x) = \bot$ then $(\gamma_1 \sqcup \gamma_2)(x) = \gamma_1(x)$ and $(\gamma_1 \sqcap \gamma_2)(x) = \bot$*
- *Otherwise, $(\gamma_1 \sqcup \gamma_2)(x) = \gamma_1(x) \sqcup \gamma_2(x)$.*

Proposition 8 (Typing Assumption Slices form Bounded Lattices)**.** *For any typing assumptions Γ, the set of slices γ of Γ form a bounded lattice with bottom element of the empty partial function $\emptyset$ and top element Γ.*

Proof. $\gamma_1(x) \neq \bot \wedge \gamma_2(x) \neq \bot$ case follows from proposition 6 and that logical conjunction distributes over forall quantifiers. The other cases follow from subsets of any set X forming bounded lattices. □

C.1.3. Expression Typing Slices

Definition 19 (Expression Typing Slices)**.** *An expression typing slice ρ is a pair ς^γ of expression slice ς and typing assumption slice γ.*

Precision

Definition 20 (Expression Typing Slice Precision)**.** *For expression typing slices $\varsigma_1^{\gamma_1}$, $\varsigma_2^{\gamma_2}$:*

$$\varsigma_1^{\gamma_1} \sqsubseteq \varsigma_2^{\gamma_2} \iff \varsigma_1 \sqsubseteq \varsigma_2 \text{ and } \gamma_1 \sqsubseteq \gamma_2$$

Proposition 9 (Precision is a Partial Order)**.** *Expression typing precision forms a partial order on expression typing slices.*

Proof. Componentwise definitions preserve universal algebras. □

Lattice Structure

Definition 21 (Expression Typing Slice Joins and Meets)**.** *For expression typing slices $\varsigma_1^{\gamma_1}$, $\varsigma_2^{\gamma_2}$:*

$$\varsigma_1^{\gamma_1} \sqcup \varsigma_2^{\gamma_2} = (\varsigma_1 \sqcup \varsigma_2)^{\gamma_1 \sqcup \gamma_2}$$
$$\varsigma_1^{\gamma_1} \sqcap \varsigma_2^{\gamma_2} = (\varsigma_1 \sqcap \varsigma_2)^{\gamma_1 \sqcap \gamma_2}$$

Proposition 10 (Expression Typing Slices for Bounded Lattices)**.** *For any expression e and typing assumption Γ, the set of expression typing slices ς^γ of e^Γ forms a bounded lattice with bottom element $\square^\emptyset$ and top element e^Γ.*

Proof. Componentwise definitions preserve universal algebras. □

Type Checking

Definition 22 (Interpreting Term Slices as Terms)**.** *By replacing gaps in terms with holes, the dynamic type, or wildcard patterns, slices can be interpreted as terms. For gaps:*

$$\frac{}{[\![\square_{typ}]\!] = ?} \qquad \frac{}{[\![\square_{pat}]\!] = \_} \qquad \frac{u \text{ is fresh}}{[\![\square_{exp}]\!] = (\!|\,|\!)^u}$$

Patterns:

$$\frac{}{[\![\_]\!] = \_} \qquad \frac{}{[\![x]\!] = x}$$

Types:

$$\frac{}{[\![\tau]\!] = \tau} \qquad \frac{[\![\upsilon_1]\!] = \tau_1 \quad [\![\upsilon_2]\!] = \tau_2}{[\![\upsilon_1 \to \upsilon_2]\!] = \tau_1 \to \tau_2}$$

Expressions:

$$\frac{}{[\![e]\!] = e} \qquad \frac{[\![p]\!] = pt \quad [\![\upsilon]\!] = \tau \quad [\![\varsigma]\!] = e}{[\![\lambda p : \upsilon.\ \varsigma]\!] = \lambda pt : \tau.\ e} \qquad \frac{[\![p]\!] = pt \quad [\![\varsigma]\!] = e}{[\![\lambda p.\ \varsigma]\!] = \lambda pt.\ e}$$

$$\frac{[\![\varsigma_1]\!] = e_1 \quad [\![\varsigma_2]\!] = e_2}{[\![\varsigma_1(\varsigma_2)]\!] = e_1(e_2)} \qquad \frac{[\![\varsigma]\!] = e \quad [\![\upsilon]\!] = \tau}{[\![\varsigma : \upsilon]\!] = e : \tau}$$

Definition 23 (Interpreting Typing Assumption slices by Typing Assumptions)**.** *Translated by extension, for typing assumption slice γ:*

$$[\![\gamma]\!](x) = [\![\gamma(x)]\!] \qquad \text{if } \gamma(x) \neq \bot$$

$$[\![\gamma]\!](x) = \bot \qquad \text{if } \gamma(x) = \bot$$

Definition 24 (Expression Typing Slice Type Checking)**.** *For expression typing slice ς^γ and type τ. $\gamma \vdash \varsigma \Rightarrow \tau$ iff $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Rightarrow \tau$ and $\gamma \vdash \varsigma \Leftarrow \tau$ iff $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Leftarrow \tau$.*

C.2. Context Typing Slices

C.2.1. Contexts

Definition 25 (Contexts Syntax)**.** *Pattern contexts – mapping patterns to patterns:*

$$\mathcal{P} ::= \bigcirc_{pat}$$

Type contexts – mapping types to types:

$$\mathcal{T} ::= \bigcirc_{typ} \mid \mathcal{T} \to \tau \mid \tau \to \mathcal{T}$$

Expression contexts – mapping patterns, types, or expression to expressions:[1]

$$\mathcal{C} ::= \bigcirc_{exp} \mid \lambda \mathcal{P} : \tau.\ e \mid \lambda pt : \mathcal{T}.\ e \mid \lambda pt : \tau.\ \mathcal{C} \mid \lambda \mathcal{P}.\ e \mid \lambda pt.\ \mathcal{C} \mid \mathcal{C}(e) \mid e(\mathcal{C}) \mid e : \mathcal{T} \mid \mathcal{C} : \tau$$

Definition 26 (Context Substitution)**.**

$$\frac{}{\bigcirc_{pat}\{pt\} = pt} \qquad \frac{}{\bigcirc_{typ}\{\tau\} = \tau} \qquad \frac{}{\bigcirc_{exp}\{e\} = e}$$

$$\frac{\mathcal{T}\{\tau_1\} = \tau_1'}{(\mathcal{T} \to \tau_2)\{\tau_1\} = \tau_1' \to \tau_2} \qquad \frac{\mathcal{T}\{\tau_2\} = \tau_2'}{(\tau_1 \to \mathcal{T})\{\tau_2\} = \tau_1 \to \tau_2'}$$

$$\frac{\mathcal{P}\{pt\} = pt'}{(\lambda \mathcal{P} : \tau.\ e)\{pt\} = \lambda pt' : \tau.\ e} \qquad \frac{\mathcal{P}\{pt\} = pt'}{(\lambda \mathcal{P}.\ e)\{pt\} = \lambda pt'.\ e}$$

[1] Note that $\mathcal{C}$ is also used for generic term contexts sometimes.

$$\frac{\mathcal{T}\{\tau\} = \tau'}{(\lambda pt : \mathcal{T}.\ e)\{\tau\} = \lambda pt : \tau'.\ e} \qquad \frac{\mathcal{T}\{\tau\} = \tau'}{(e : \mathcal{T})\{\tau\} = e : \tau'}$$

$$\frac{\mathcal{C}\{e\} = e'}{(\lambda pt : \tau.\ \mathcal{C})\{\tau\} = \lambda pt : \tau.\ e'} \qquad \frac{\mathcal{C}\{e\} = e'}{(\lambda pt.\ \mathcal{C})\{\tau\} = \lambda pt.\ e'} \qquad \frac{\mathcal{C}\{e_1\} = e_1'}{(\mathcal{C}(e_2))\{e_1\} = e_1'(e_2)}$$

$$\frac{\mathcal{C}\{e_2\} = e_2'}{(e_1(\mathcal{C}))\{e_2\} = e_1(e_2')} \qquad \frac{\mathcal{C}\{e\} = e'}{(\mathcal{C} : \tau)\{e\} = e' : \tau}$$

The input and output classes of contexts $\mathcal{C}$ will be notated $\mathcal{C} : X \to Y$ for `Pat` (patterns), `Typ` (types), `Exp` (expressions).

Definition 27 (Context Composition)**.** *Defined analogously as context substitution, but substituting contexts syntactically, provided the input and output classes match. If $\mathcal{C}_1 : X \to Y$ and $\mathcal{C}_2 : Y \to Z$ then $\mathcal{C}_2\{C_1\} = \mathcal{C}_2 \circ \mathcal{C}_1 : X \to Z$. Equivalence of contexts can be defined syntactically and coincides exactly with an extensional definition.*

Composition could alternatively be defined pointwise, and we get the same result here, but it is later important we keep this syntactic to allow decomposition around the marks inside contexts.

Proposition 11 (Context composition is associative)**.** *For all $\mathcal{C}_1 : X \to Y$, $\mathcal{C}_2 : Y \to Z$, and $\mathcal{C}_3 : Z \to W$ then:*

$$(\mathcal{C}_3 \circ \mathcal{C}_2) \circ C_1 = \mathcal{C}_3 \circ (\mathcal{C}_2 \circ C_1)$$

Proof. Trivial. □

C.2.2. Context Slices

Syntax extended analogously to term slices. Use c to represent context slices.

Precision

Definition 28 (Context Precision)**.** *If $c : X \to Y$ and $c' : X \to Y$ are context slices, then $c' \sqsubseteq c$ if and only if, for all terms t of class X, that $c'\{t\} \sqsubseteq c\{t\}$.*

Proposition 12 (Precision is a Partial Order)**.** *Context precision forms a partial order on context slices.*

Proof. Follows from $\sqsubseteq$ on term slices being a partial order, and that logical conjunction distributes over forall quantifiers. □

Proposition 13 (Context Filling Preserves Precision)**.** *For context slice $c : X \to Y$ and term slice ς of class X. Then if we have slices $\varsigma' \sqsubseteq \varsigma$, $c' \sqsubseteq c$ then also $c'\{\varsigma'\} \sqsubseteq c\{\varsigma\}$.*

Proof. $c'\{\varsigma\} \sqsubseteq c\{\varsigma\}$ follows directly from definition of context precision. $c\{\varsigma'\} \sqsubseteq c\{\varsigma\}$ is due substituted contexts retaining their inputs as sub-terms. □

Lattice Structure

Definition 29 (Context Slice Joins & Meets)**.** *For context slices $c_1 : X \to Y$ and $c_2 : X \to Y$ and any term t of class X:*

$$(c_1 \sqcup c_2)\{t\} = c_1\{t\} \sqcup c_2\{t\}$$

$$(c_1 \sqcap c_2)\{t\} = c_1\{t\} \sqcap c_2\{t\}$$

Definition 30 (Purely Structural Contexts)**.** *Least specific slices of some context, containing only gaps $\square$ and the mark $\bigcirc$. The purely structural context of $\mathcal{C}$ is the unique one that is a slice of $\mathcal{C}$*

$$\mathcal{P}_s ::= \bigcirc_{pat}$$

$$\mathcal{T}_s ::= \bigcirc_{typ} \mid \mathcal{T}_s \to \square \mid \square \to \mathcal{T}$$

$$\mathcal{C}_s ::= \bigcirc_{exp} \mid \lambda \mathcal{P} : \square.\ \square \mid \lambda \square : \mathcal{T}.\ \square \mid \lambda \square : \square.\ \mathcal{C} \mid \lambda \mathcal{P}.\ \square \mid \lambda \square.\ \mathcal{C} \mid \mathcal{C}(\square) \mid \square(\mathcal{C}) \mid \square : \mathcal{T} \mid \mathcal{C} : \square$$

Proposition 14 (Context Slices form Bounded Lattices). *For any context $\mathcal{C}$, the set of slices c of $\mathcal{C}$ form a bounded lattice with bottom element of the purely structural context of $\mathcal{C}$ and top element $\mathcal{C}$.*

Proof. Follows smoothly as usual. Bottom element is different for each bounded lattice in this case. □

C.2.3. Typing Assumption Contexts & Context Slices

Definition 31 (Typing Assumption Contexts & Slices). *A typing assumption context $\mathcal{F}$ is a function from typing assumption to typing assumptions. A typing assumption context slice f is a function from typing assumption slices to typing assumption slices.*

Precision

Definition 32 (Typing Assumption Context Slice Precision). *If f' and f are typing assumption context slices, then $f' \sqsubseteq f$ if and only if, for all typing context slices γ, that $f'(\gamma) \sqsubseteq f(\gamma)$.*

Proposition 15 (Precision is a Partial Order). *Typing assumption context precision forms a partial order on typing assumption context slices.*

Proof. Pointwise definitions preserve this structure as before. □

Proposition 16 (Function Application Preserves Precision). *For typing assumption slice γ and typing assumption context slice f. Then if we have slices $\gamma' \sqsubseteq \gamma$, $f' \sqsubseteq f$ then also $f'(\gamma') \sqsubseteq f(\gamma)$.*

Lattice Structure

Definition 33 (Typing Assumption Context Slice Joins & Meets). *For typing assumption context slices f_1 and f_2 and any typing assumption slice γ:*

$$(f_1 \sqcup f_2)(\gamma) = f_1(\gamma) \sqcup f_2(\gamma)$$

$$(f_1 \sqcap f_2)(\gamma) = f_1(\gamma) \sqcap f_2(\gamma)$$

Proposition 17 (Typing Assumption Context Slices form Bounded Lattices). *For any typing assumption context $\mathcal{F}$, the set of slices f of $\mathcal{F}$ form a bounded lattice with bottom element being the constant function to the empty typing assumption function and top element $\mathcal{F}$.*

Proof. Pointwise definitions preserve this structure as before. □

C.2.4. Context Typing Slices

Definition 34 (Expression Context Typing Slice). *An expression context typing slice $\mathfrak{p}$ is a pair context slice with sub-contexts recursively tagged by typing assumption context slice f. If f is omitted, then assume it is the identity.*

Precision

Definition 35 (Expression Context Typing Slice Precision). *For expression context typing slices $c_1^{f_1}$, $c_2^{f_2}$:*

$$c_1^{f_1} \sqsubseteq c_2^{f_2} \iff c_1 \sqsubseteq c_2 \text{ and } f_1 \sqsubseteq f_2$$

Proposition 18 (Precision is a Partial Order). *Typing assumption context precision forms a partial order on typing assumption context slices.*

Proof. Componentwise definitions preserve universal algebras. □

Composition

Definition 36 (Expression Context Typing Slice Composition)**.** *It is important that we define composition* $c_1^{f_1} \circ c_1^{f_2}$ ***syntactically****. That is, the same definition as context substitution but with each sub-context tagged with typing assumption contexts.*

Proposition 19 (Expression Context Slice Composition is Associative)**.** *For all* p_1*,* p_2*, and* p_3 *then:*

$$(p_3 \circ p_2) \circ p_1 = p_3 \circ (p_2 \circ p_1)$$

Proof. Trivial. □

Definition 37 (Expression Context Typing Slice Substitution)**.**

$$\frac{}{\bigcirc^{\mathcal{F}}\{t^{\gamma}\} = t^{\mathcal{F}(\gamma)}}$$

$$\frac{\mathcal{C}_2^{\mathcal{F}}\{e_2^{\gamma_2}\} = e_2'^{\gamma_2'}}{(e_1(\mathcal{C}^{\mathcal{F}_2}))^{\mathcal{F}_1}\{e_2^{\gamma_2}\} = (e_1(e_2'))^{\mathcal{F}_1\{\gamma_2'\}}} \qquad \frac{\mathcal{C}_1^{\mathcal{F}}\{e^{\gamma}\} = e'^{\gamma'}}{(\mathcal{C}^{\mathcal{F}_1} : \tau)^{\mathcal{F}_2}\{e^{\gamma}\} = (e' : \tau)^{\mathcal{F}_2\{\gamma'\}}}$$

etc. (similarly extended to all terms)

Lattice Structure

Definition 38 (Expression Context Typing Slice Joins and Meets)**.** *For expression typing slices* $c_1^{f_1}$*,* $c_2^{f_2}$*:*

$$c_1^{f_1} \sqcup c_2^{f_2} = (c_1 \sqcup c_2)^{f_1 \sqcup f_2}$$
$$c_1^{f_1} \sqcap c_2^{f_2} = (c_1 \sqcap c_2)^{f_1 \sqcap f_2}$$

Proposition 20 (Expression Context Typing Slices form Bounded Lattices)**.** *For any expression context* $\mathcal{C}$ *and typing assumption context* $\mathcal{F}$*, the set of expression context typing slices* c^f *of* $\mathcal{C}^{\mathcal{F}}$ *forms a bounded lattice with bottom element, the purely structural context and the function to the empty typing assumptions, and top element* e^{Γ}*.*

Proof. Componentwise definitions preserve universal algebras. □

Interpreting Context Typing Slices by Contexts and Typing Assumption Contexts

A $[\![c]\!]$ function can be analogously defined as to expressions, replacing the gaps by $\_$, ?, $(\!|\,|\!)^u$.

C.3. Type-Indexed Slices

C.3.1. Type-Indexed Context Typing Slices

Definition 39 (Type-Indexed Context Typing Slices)**.** *Syntactically defined:*

$$\mathcal{S} ::= p \mid p * \mathcal{S} \to p * \mathcal{S}$$

With any $\mathcal{S}$ *only being valid if it has a full slice. The full slice of* $\mathcal{S}$ *is notated* $\overline{\mathcal{S}}$ *and defined recursively:*

$$\overline{p} = p$$
$$\overline{p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2} = p_1 \circ \overline{\mathcal{S}_1} \sqcup p_2 \circ \overline{\mathcal{S}_2}$$

Definition 40 (Type-Indexed Context Typing Slice Composition)**.** *For type-indexed context typing slices* $\mathcal{S}$ *and* $\mathcal{S}'$*. If* $\mathcal{S} = p$ *and* $\mathcal{S}' = p'$*:*

$$p' \circ p = \overline{p'} \circ \overline{p} \qquad p \circ p' = \overline{p} \circ \overline{p'}$$

If $\mathcal{S} = p$ *and* $\mathcal{S}' = p_1' * \mathcal{S}_1' \to p_2' * \mathcal{S}_2'$*:*

$$S \circ \mathcal{S}' = (p \circ p_1') * \mathcal{S}_1' \to (p \circ p_2') * \mathcal{S}_2'$$

If $\mathcal{S} = p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2$*:*

$$\mathcal{S} \circ \mathcal{S}' = p_1 * (\mathcal{S}_1 \circ \mathcal{S}') \to p_2 * (\mathcal{S}_2 \circ \mathcal{S}')$$

Proposition 21 (Type-Indexed Composition Preserves Full Slice Composition)**.** *For type-indexed slices $\mathcal{S}$ and $\mathcal{S}'$:*

$$\overline{\mathcal{S} \circ \mathcal{S}'} = \overline{\mathcal{S}} \circ \overline{\mathcal{S}'}$$

Proof. Trivial induction on the structure of $\mathcal{S}$. □

C.3.2. Type-Indexed Expression Typing Slices

(Overloading the $\mathcal{S}$ notation)

Definition 41 (Type-Indexed Expressions Typing Slices)**.** *Syntactically defined:*

$$\mathcal{S} ::= \rho \mid p * \mathcal{S} \to p * \mathcal{S}$$

With any $\mathcal{S}$ only being valid if it has a full slice. The full slice of $\mathcal{S}$ is notated $\overline{\mathcal{S}}$ and defined recursively:

$$\overline{\rho} = \rho$$
$$\overline{p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2} = p_1\{\overline{\mathcal{S}_1}\} \sqcup p_2\{\overline{\mathcal{S}_2}\}$$

We retain left incremental composition/application as before (but applying to leaves ρ) but do not retain global right composition.

C.3.3. Function Matching

Definition 42 (Slice Function Matching)**.**

$$p \blacktriangleright_{\to} \bigcirc * p \to \bigcirc * p \qquad \rho \blacktriangleright_{\to} \bigcirc * \rho \to \bigcirc * \rho$$

$$p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2 \blacktriangleright_{\to} p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2$$

C.3.4. Global Application

Global application and reverse application can be defined to allow converting between context and expression slices.

Definition 43 (Reverse Application)**.** *For an expressions slice ρ. Reverse application $|>$ converts type-indexed context typing slices to type-indexed expressions typing slices:*

$$\rho \ |> \ p = p(\rho)$$

$$\rho \ |> \ (p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2) = p_1 * (\rho \ |> \ \mathcal{S}_1) \to p_2 * (\rho \ |> \ \mathcal{S}_2)$$

Proposition 22 (Validity)**.** *If $\mathcal{S}$ is a valid type indexed context typing slice. For all expression typing slices ρ, then $\rho \ |> \ \mathcal{S}$ is a valid type-indexed expression typing slice with the same structure as $\mathcal{S}$.*

Proof. Trivial induction on the structure of $\mathcal{S}$. □

Definition 44 (Application)**.** *For a function f from expression typing slices ρ to context typing slices. Application $\$$ converts type-indexed expression typing slices to type-indexed context typing slices:*

$$f \ \$ \ \rho = f(\rho)$$

$$f \ \$ \ (p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2) = \bigcirc * (f \ \$ \ p_1 \circ \mathcal{S}_1) \to \bigcirc * (f \ \$ \ p_2 \circ \mathcal{S}_2)$$

The contexts p_1, p_2 are eagerly applied down to the leaves ρ, to ensure the validity property.

Proposition 23 (Validity)**.** *If $\mathcal{S}$ is a valid type indexed expression typing slice. For all functions f from expression slices to context slices, then $f \ \$ \ \mathcal{S}$ is a valid type-indexed context typing slice with the same structure as $\mathcal{S}$.*

Proof. Trivial induction on the structure of $\mathcal{S}$. □

C.4. Checking Contexts

Definition 45 (Checking Context). *For term e checking against τ: $\Gamma \vdash e \Leftarrow \tau$. A checking context for e is an expression context $\mathcal{C}$ and typing assumption context $\mathcal{F}$ such that:*

- $\mathcal{C} \neq \bigcirc$.
- $\mathcal{F}(\Gamma) \vdash \mathcal{C}\{e\} \Rightarrow \tau'$ *for some* τ'.
- *The above derivation has a sub-derivation* $\Gamma \vdash e \Leftarrow \tau$.

Definition 46 (Minimally Scoped Checking Context). *For a derivation $\Gamma \vdash e \Leftarrow \tau$, a minimally scoped expression checking context is a checking context of e such that no sub-context is also a checking context.*

Proposition 24 (Minimally Scoped Checking Contexts Forms). *All minimally scoped contexts, have the following forms. Defined by a judgement: $\Gamma \vdash \mathcal{C}^{\mathcal{F}}$ checks e against τ. With the meaning: $\mathcal{C}^{\mathcal{F}}$ is a minimally scoped checking context for $\Gamma \vdash e \Leftarrow \tau$. Defined:*

$$\frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash \bigcirc : \tau \text{ checks } e \text{ against } \tau} \qquad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \quad \Gamma \vdash e_2 \Leftarrow \tau}{\Gamma \vdash e_1(\bigcirc) \text{ checks } e_2 \text{ against } \tau_2}$$

$$\frac{\Gamma(x) = \tau_1 \quad \Gamma \backslash x : \tau_1 \vdash \mathcal{C}^{\mathcal{F}} \text{ checks } \lambda x.\ e \text{ against } \tau_1 \to \tau_2}{\Gamma \vdash \mathcal{C}^{\mathcal{F}} \circ (\lambda x.\ \bigcirc)^{\Gamma \mapsto \Gamma \backslash x:\tau_1} \text{ checks } e \text{ against } \tau_2}$$

Proof. Verify that each rule is a checking context, this follows very directly from the Hazel typing rules. No rule has a sub-context also being a checking context by induction, with the base cases being trivial: there is only one sub-context for the base case rules, $\bigcirc$, which is by definition not a checking context. □

Proposition 25 (Checking Context of a Sub-term in a Derivation). *If derivation $\Gamma \vdash e \Rightarrow \tau$ contains a analysis sub-derivation $\Gamma' \vdash e' \Leftarrow \tau'$ for sub-term e'. Then there is a unique minimally scoped context $\mathcal{C}$ for e' and typing assumption context $\mathcal{F}$ for Γ' such that:*

- $\mathcal{C}\{e'\}$ *is a sub-term of e, or is e itself.*
- *Derivation $\Gamma \vdash e \Rightarrow \tau$ contains a sub-derivation for $\mathcal{F}(\Gamma') \vdash \mathcal{C}\{e'\} \Rightarrow \tau''$, or is the derivation itself: $\tau'' = \tau, \mathcal{C}\{e'\} = e$, and $\mathcal{F}(\Gamma') = \Gamma$.*

Proof. Every minimally scoped checking context for e' has a different structure. Hence, only one of these matches with the structure of e' in e. You simply need to verify that one always exists (induction on the typing derivation), and that $\mathcal{F}(\Gamma') = \Gamma$ when $\mathcal{C}(e') = e$. □

C.5. Criterion 1: Synthesis Slices

Definition 47 (Synthesis Slices). *For a synthesising expression, synthesiseτ. A synthesis slice is an expression typing slice ς^{γ} of e^{Γ} which also synthesises τ, that is, $[\![\gamma]\!] \vdash [\![\varsigma]\!] \Rightarrow \tau$.*

Proposition 26 (Minimum Synthesis Slices). *A minimum synthesis slice of e^{Γ} is a synthesis slice ρ such that any other synthesis slices ρ' are at least as specific, $\rho \sqsubseteq \rho'$. This minimum always uniquely exists.*

Proof. Existence follows from bounded lattice structure, uniqueness follows from the uniqueness of typing in Hazel. □

Definition 48 (Type-Indexed Slices of Types).

$$\widehat{b} = b^{\emptyset}$$

$$\widehat{\tau_1 \to \tau_2} = (\bigcirc \to \square) * \hat{(}\tau_1) \to (\square \to \bigcirc) * \widehat{\tau_2}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau \mid \mathcal{S}}$ e synthesising type τ under context Γ produces minimum type-indexed synthesis slice $\mathcal{S}$

$$\text{SConst}\ \frac{}{\Gamma \vdash c \Rightarrow b \mid c^{\emptyset}} \qquad \text{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \mid x^{x:\tau}} \qquad \text{SVar?}\ \frac{x : ? \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \mid \square^{\emptyset}}$$

$$\text{SFun}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \mid \mathcal{S}_2 \qquad [\![S_2]\!] = \varsigma^{\gamma} \qquad \gamma(x) = \tau_1'}{\Gamma \vdash \lambda x : \tau_1.\ e \Rightarrow \tau_1 \to \tau_2 \mid (\lambda \square : \bigcirc.\ \square)^{\gamma \mapsto \gamma \setminus x:\tau_1} * \widehat{\tau_1} \to (\lambda x : \tau_1'.\ \bigcirc)^{\gamma \mapsto \gamma \setminus x:\tau_1'} * \mathcal{S}_2}$$

$$\text{SFunUnused}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \mid \mathcal{S}_2 \qquad [\![S_2]\!] = \varsigma^{\gamma} \qquad \gamma(x) = \bot}{\Gamma \vdash \lambda x : \tau_1.\ e \Rightarrow \tau_1 \to \tau_2 \mid (\lambda \square : \bigcirc.\ \square)^{\gamma \mapsto \gamma \setminus x:\tau_1} * \widehat{\tau_1} \to (\lambda \square : \square.\ \bigcirc) * \mathcal{S}_2}$$

$$\text{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \mid \mathcal{S}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \mid (\bigcirc(\square)) \circ \mathcal{S}} \qquad \text{SEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^{u} \Rightarrow ? \mid \square^{\emptyset}}$$

$$\text{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^{u} \Rightarrow ? \mid \square^{\emptyset}} \qquad \text{SAsc}\ \frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau \mid (\square : \bigcirc) \circ \widehat{\tau}}$$

Figure C.1: *Minimum synthesis slice* calculation

Conjecture 2 (Correctness)**.** *If* $\Gamma \vdash e \Rightarrow \tau$ *then:*

- $\Gamma \vdash e \Rightarrow \tau \mid \rho$ *where* $\rho = \varsigma^{\gamma}$ *with* $\gamma \vdash \varsigma \Rightarrow \tau$.
- *For any* $\rho' = \varsigma'^{\gamma'} \sqsubseteq e^{\Gamma}$ *such that* $\gamma' \vdash \varsigma' \Rightarrow \tau$ *then* $\rho \sqsubseteq \rho'$.

C.6. Criterion 2: Analysis Slices

Definition 49 (Analysis Slice)**.** *For a term* e *analysing against* τ*:* $\Gamma \vdash e \Leftarrow \tau$*, and a minimally scoped checking context* $\mathcal{C}^{\mathcal{F}}$ *for* e*. An analysis slice is a slice* $c^{\prime}$ *of* $\mathcal{C}^{\mathcal{F}}$ *which is also a checking context for* e*. That is:*

- $f(\Gamma) \vdash c\{e\} \Rightarrow \tau'$ *for some* τ'.
- *The above derivation has a sub-derivation for* $\Gamma \vdash e \Leftarrow \tau$.

Conjecture 3 (Minimum Analysis Slices)**.** *The minimum analysis slice analysing* e *in a checking context* $\mathcal{C}^{\mathcal{F}}$ *is an analysis slice* p *such that for any other any other analysis slice* p' *is at least as specific,* $p \sqsubseteq p'$*. This minimum always uniquely exists.*

$\boxed{\Gamma; \mathcal{C} \vdash e \Leftarrow \tau \mid \mathcal{S}}$ e which analyses against type τ in minimally scoped checking context $\mathcal{C}$ and typing context Γ has minimum type-indexed analysis slice $\mathcal{S}$.

$$\text{AAnnot}\ \frac{\Gamma \vdash (\bigcirc : \tau)^{\mathcal{F}}\ \text{checks}\ e\ \text{against}\ \tau}{\Gamma; (\bigcirc : \tau)^{\mathcal{F}} \vdash e \Leftarrow \tau \mid (\varsigma^{\gamma} \mapsto (\bigcirc : \varsigma)^{\gamma})\ \$\ \widehat{\tau}}$$

$$\text{AApp}\ \frac{\Gamma \vdash (e_1(\bigcirc))^{\mathcal{F}}\ \text{checks}\ e_2\ \text{against}\ \tau_2 \qquad \mathcal{F}(\Gamma) \vdash e_1 \Rightarrow \tau_1 \mid \mathcal{S}_1 \qquad \mathcal{S}_1 \blacktriangleright_{\to} p_2 * \mathcal{S}_2 \to p * \mathcal{S}}{\Gamma; (e_1(\bigcirc))^{\mathcal{F}} \vdash e_2 \Leftarrow \tau_2 \mid (\varsigma^{\gamma} \mapsto (\varsigma(\bigcirc))^{\gamma' \mapsto \gamma' \sqcup \gamma})\ \$\ p_2 \circ \mathcal{S}_2}$$

$$\text{AFun}\ \frac{\Gamma(x) = \tau_1 \qquad \mathcal{F}_1(\Gamma); \mathcal{C}^{\mathcal{F}_2} \vdash \lambda x.\ e \Leftarrow \tau_1 \to \tau_2 \mid \mathcal{S} \qquad \mathcal{S} \blacktriangleright_{\to} p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2}{\Gamma; \mathcal{C}^{\mathcal{F}_2} \circ (\lambda x.\ \bigcirc)^{\mathcal{F}_1} \vdash e \Leftarrow \tau_2 \mid p_2 \circ \mathcal{S}_2 \circ (\lambda \square.\ \bigcirc)^{\gamma \mapsto \gamma \setminus x:\tau_1}}$$

As composition has been defined **syntactically**, then $\mathcal{C}^{\mathcal{F}_2} \circ (\lambda x.\ \bigcirc)^{\mathcal{F}_1}$, is a valid way to pattern match upon contexts, hence contexts for AFun can be deconstructed uniformly in such a way.

Figure C.2: *Minimum analysis slice* calculation

Issues with checking equality of functions might arise in implementing this. But, typing assumption contexts can be made intensional by assuming finite domain. It is probably also possible to remove function equality checks entirely.

Conjecture 4 (Correctness). *If $\mathcal{C}$ is a checking context for e and $\Gamma \vdash e \Leftarrow \tau$. Then $\Gamma; \mathcal{C} \vdash e \Leftarrow \tau \mid \mathcal{S}$ and $\mathcal{S}$ is the minimum analysis slice of e.*

C.7. Criterion 3: Contribution Slices

Definition 50 (Contribution Slices). *For $\Gamma \vdash e \Rightarrow \tau$ containing sub-derivation $\Gamma' \vdash e' \Leftarrow \tau'$ with checking context $\mathcal{C}$.*

A contribution slice of e' is an analysis slice for e' in $\mathcal{C}$ paired with an expression typing slice ς^{γ} such that:

- *ς is a slice of e', that $\varsigma \sqsubseteq e'$.*
- *Under restricted typing context γ, that ς checks against any τ_2' at least as precise as τ':*[2]

$$\forall \tau_2'.\ \tau' \sqsubseteq \tau_2' \implies \gamma \vdash e' \Leftarrow \tau_2'$$

A contribution slice for a sub-term e'' involved in sub-derivation $\Gamma'' \vdash e'' \Rightarrow \tau''$ where $e'' \neq e'$ is an expression typing slice $\varsigma''^{\gamma''}$ which also synthesises τ'' under γ'', that $\gamma'' \vdash \varsigma'' \Rightarrow \tau''$. Further, any sub-term of e'' which has a contribution slice of the above variety, is replaced inside ς by that corresponding expression typing slice.

The synthetic parts of these slices can be calculated in exactly the same way as synthesis slices, except now also considering the *subsumption* rule. The analytic parts are regular analysis slices.

The subsumption rule synthesises a type for some term, then checks consistency with the checked type. This is where the dynamic portions can be omitted, to give a contribution slice for the checked term. Representing contribution slices with the judgement $\Gamma \vdash e \Rightarrow \tau \mid_{\mathrm{C}} \mathcal{S}$ and $\Gamma; \mathcal{C} \vdash e \Leftarrow \tau \mid_{\mathrm{C}} \mathcal{S}_e \mid \mathcal{S}_c$, where $\mathcal{S}_e$ is the expression slice part and $\mathcal{S}_c$ is the contextual part:

$$\frac{\Gamma \vdash e \Rightarrow \tau \mid_{\mathrm{C}} \mathcal{S}_s \qquad \tau \sim \tau' \qquad \Gamma; \mathcal{C} \vdash e \Leftarrow \tau' \mid \mathcal{S}_c}{\Gamma; \mathcal{C} \vdash e \Leftarrow \tau' \mid_{\mathrm{C}} \mathrm{match}(\mathcal{S}_c, \mathcal{S}_s) \mid \mathcal{S}_c}$$

Where match takes the matches the *right* slice to have the same structure as the *left slice*, expanding with funmatching if required:

$$\mathrm{match}(p, p') = p'$$

$$\mathrm{match}(p, \_ \to \_) = \square^{\emptyset}$$

$$\mathrm{match}(\_ \to \_, p) = \bigcirc * p \to \bigcirc * p$$

$$\mathrm{match}(c_1 * \mathcal{S}_1 \to c_2 * \mathcal{S}_2,\ c_1' * \mathcal{S}_1' \to c_2' * \mathcal{S}_2') = c_1 * \mathrm{match}(\mathcal{S}_1, \mathcal{S}_1') \to c_2 * \mathrm{match}(\mathcal{S}_2, \mathcal{S}_2')$$

C.8. Cast Syntax

The below demonstrates inserting synthesis and analysis slices, but contribution slices could equally be used instead. Extend the cast syntax to casts between types and synthesis and or analysis slices. The pair of slices explain why the cast was enforced, and they will always match structurally with the annotated type.

To avoid conflict, analysis (contextual) slices are now notated $\mathcal{A}$. Generally speaking a cast will be *from* synthesised types *to* analysed expectations.

$$\langle \tau_1 \mid \mathcal{S}_1 \mid \mathcal{A}_1 \Rightarrow \tau_2 \mid \mathcal{A}_2 \mid \mathcal{S}_2 \rangle$$

If an analysis slice or synthesis slice are omitted, assume they are $\bigcirc^{\emptyset}$ or $\square^{\emptyset}$ respectively.

[2] Essentially, sub-terms that check against ? also synthesise ?. Defined this way to include the case of unannotated lambdas (which do not synthesise).

C.9. Elaboration

Nicely complements existing elaboration rules (fig. C.3).

Note that elaboration adds a new situation analysing left application parts. The ESApp rule is split explicitly into the cases when $\tau_1 = ?$, as the cast on d_1 need only be inserted in this case. We are effectively adding in a new analysis slice for e_1 in $\bigcirc(e_2)$ when e_1 synthesises $?$. The fact it is a function is then derived from the outer context:

Definition 51 (Dynamic Synthesising Function Analysis Slice)**.**

$$\frac{\Gamma \vdash e_1 \Rightarrow ?}{\Gamma \vdash \bigcirc(e_2) \ \mathit{checks}\ e_1 \ \mathit{against}\ \tau_2 \to \tau}$$

$$\mathit{AFunDyn}\ \frac{\Gamma \vdash (\bigcirc : \tau)^{\mathcal{F}} \ \mathit{checks}\ e \ \mathit{against}\ \tau}{\Gamma; (\bigcirc(e_2))^{\mathcal{F}} \vdash e_1 \Leftarrow \tau_2 \to \tau \mid \bigcirc * (\bigcirc(\square))^{\mathcal{F}} \to \bigcirc * (\bigcirc(\square))^{\mathcal{F}}}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau \mid \mathcal{S} \rightsquigarrow d \dashv \Delta}$ e synthesises type τ with synthesis slice $\mathcal{S}$ and elaborates to d and hole context Δ.

$$\text{ESConst}\ \frac{}{\Gamma \vdash c \Rightarrow b \mid c^{\emptyset} \rightsquigarrow c \dashv \emptyset} \qquad \text{ESVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \mid x^{x:\tau} \rightsquigarrow x \dashv \emptyset}$$

$$\text{ESFunUnused}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \mid \mathcal{S}_2 \rightsquigarrow d \dashv \Delta \qquad [\![S_2]\!] = \varsigma^{\gamma} \qquad \gamma(x) = \bot}{\Gamma \vdash \lambda x : \tau_1.e \Rightarrow \tau_1 \to \tau_2 \mid (\lambda\square : \bigcirc.\ \square)^{\gamma \mapsto \gamma \backslash x:\tau_1} * \widehat{\tau_1} \to (\lambda\square : \square.\ \bigcirc) * \mathcal{S}_2 \rightsquigarrow \lambda x : \tau_1.d \dashv \Delta}$$

$$\text{ESFun}\ \frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \mid \mathcal{S}_2 \rightsquigarrow d \dashv \Delta \qquad [\![S_2]\!] = \varsigma^{\gamma} \qquad \gamma(x) = \tau_1'}{\Gamma \vdash \lambda x : \tau_1.e \Rightarrow \tau_1 \to \tau_2 \mid (\lambda\square : \bigcirc.\ \square)^{\gamma \mapsto \gamma \backslash x:\tau_1} * \widehat{\tau_1} \to (\lambda x : \tau_1'.\ \bigcirc)^{\gamma \mapsto \gamma \backslash x:\tau_1'} * \mathcal{S}_2 \rightsquigarrow \lambda x : \tau_1.d \dashv \Delta}$$

$$\text{ESApp}\ \frac{\begin{array}{ccc}\Gamma \vdash e_1 \Rightarrow \tau_1 \mid \mathcal{S}_1 & \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau & \mathcal{S}_1 \blacktriangleright_{\to} p_2 * \mathcal{S}_2 \to p * \mathcal{S} \\ \Gamma;\ \bigcirc(e_2) \vdash e_1 \Leftarrow \tau_2 \to \tau \mid \mathcal{A}_1 \rightsquigarrow d_1 : \tau_1' \mid \mathcal{S}_1' \mid \mathcal{A}_1' \dashv \Delta_1 & \Gamma;\ e_1(\bigcirc) \vdash e_2 \Leftarrow \tau_2 \mid \mathcal{A}_2 \rightsquigarrow d_2 : \tau_2' \mid \mathcal{S}_2' \mid \mathcal{A}_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \mid \mathcal{S}_1' \mid \mathcal{A}_1' \Rightarrow \tau_2 \to \tau \mid \mathcal{A}_1 \mid p_2 * \mathcal{S}_2 \to p * \mathcal{S}\rangle)(d_2\langle \tau_2' \mid \mathcal{S}_2' \mid \mathcal{A}_2' \Rightarrow \tau_2 \mid \mathcal{A}_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\text{ESEHole}\ \frac{}{\Gamma \vdash (\!|)^u \Rightarrow ? \rightsquigarrow \square^{\emptyset} \dashv (\!|)^u_{\mathrm{id}(\Gamma)} u :: (\!|)[\Gamma]}$$

$$\text{ESNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau \mid \mathcal{S} \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Rightarrow ? \mid \square^{\emptyset} \rightsquigarrow (\!|d|\!)^u_{\mathrm{id}(\Gamma)} \dashv \Delta, u :: (\!|)[\Gamma]}$$

$$\text{ESAsc}\ \frac{\Gamma;\ \bigcirc : \tau \vdash e \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow d : \tau' \mid \mathcal{S}' \mid \mathcal{A}' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \mid (\square : \bigcirc) \circ \widehat{\tau} \rightsquigarrow d\langle \tau' \mid \mathcal{S}' \mid \mathcal{A}' \Rightarrow \tau \mid \mathcal{A}\rangle \dashv \Delta}$$

$\boxed{\Gamma;\ \mathcal{C}^{\mathcal{F}} \vdash e \Leftarrow \tau \mid d \rightsquigarrow \tau' : \mathcal{S}' \mid \mathcal{A}' \mid \Delta \dashv}$ e in minimally scoped checking context $\mathcal{C}^{\mathcal{F}}$ analyses against type τ with analysis slice $\mathcal{A}$ and elaborates to d of consistent type τ' which has synthesis slice part $\mathcal{S}'$ and analysis slice part $\mathcal{A}'$ (structurally matching the synthetic part).

$$\text{EAFun}\ \frac{\begin{array}{ccc}\Gamma; \mathcal{C} \vdash \lambda x.\ e \Leftarrow \tau \mid \mathcal{A} & \tau \blacktriangleright_{\to} \tau_1 \to \tau_2 & \mathcal{A} \blacktriangleright_{\to} p_1 * \mathcal{A}_1 \to p_2 * \mathcal{A}_2 \\ \Gamma, x : \tau_1;\ \mathcal{C} \circ (\lambda x.\ \bigcirc)^{\Gamma \mapsto \Gamma \backslash x:\tau_1} \vdash e \Leftarrow \tau_2 \mid \mathcal{A}_2 \rightsquigarrow d : \tau_2' \mid \mathcal{S}_2' \mid \mathcal{A}_2' \dashv \Delta & & \\ [\![\mathcal{S}_2']\!] = \varsigma^{\gamma} & \gamma(x) = \tau_1' & \mathcal{A}_2' = \mathcal{A}_2'' \circ (\lambda pt.\ \bigcirc)^{\mathcal{F}}\end{array}}{\Gamma;\ \mathcal{C} \vdash \lambda x.e \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow \lambda x : \tau_1.d : \tau_1 \to \tau_2' \mid \bigcirc * \square^{\emptyset} \to (\lambda x.\ \bigcirc)^{\gamma \to \gamma \backslash x:\tau_1'} * \mathcal{S}_2' \mid p_1 * \mathcal{A}_1 \to \bigcirc * \mathcal{A}_2'' \dashv \Delta}$$

$$\text{EAFunUnused}\ \frac{\begin{array}{ccc}\Gamma; \mathcal{C} \vdash \lambda x.\ e \Leftarrow \tau \mid \mathcal{A} & \tau \blacktriangleright_{\to} \tau_1 \to \tau_2 & \mathcal{A} \blacktriangleright_{\to} p_1 * \mathcal{A}_1 \to p_2 * \mathcal{A}_2 \\ \Gamma, x : \tau_1;\ \mathcal{C} \circ (\lambda x.\ \bigcirc)^{\Gamma \mapsto \Gamma \backslash x:\tau_1} \vdash e \Leftarrow \tau_2 \mid \mathcal{A}_2 \rightsquigarrow d : \tau_2' \mid \mathcal{S}_2' \mid \mathcal{A}_2' \dashv \Delta & & \\ [\![\mathcal{S}_2']\!] = \varsigma^{\gamma} & \gamma(x) = \bot & \mathcal{A}_2' = \mathcal{A}_2'' \circ (\lambda pt.\ \bigcirc)^{\mathcal{F}}\end{array}}{\Gamma;\ \mathcal{C} \vdash \lambda x.e \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow \lambda x : \tau_1.d : \tau_1 \to \tau_2' \mid \bigcirc * \square^{\emptyset} \to (\lambda\square.\ \bigcirc) * \mathcal{S}_2' \mid p_1 * \mathcal{A}_1 \to \bigcirc * \mathcal{A}_2'' \dashv \Delta}$$

$$\text{EASubsume}\ \frac{\begin{array}{c}e \neq (\!|)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma; \mathcal{C} \vdash e \Leftarrow \tau \mid \mathcal{A} \\ \Gamma \vdash e \Rightarrow \tau' \mid \mathcal{S}' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'\end{array}}{\Gamma;\ \mathcal{C} \vdash e \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow d : \tau' \mid \mathcal{S}' \mid \mathrm{match}(\mathcal{S}', \mathcal{A}) \dashv \Delta}$$

$$\text{EAEHole}\ \frac{\Gamma; \mathcal{C} \vdash (\!|)^u \Leftarrow \tau \mid \mathcal{A}}{\Gamma;\ \mathcal{C} \vdash (\!|)^u \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow (\!|)^u_{\mathrm{id}(\Gamma)} : \tau \mid \mathrm{match}(\mathcal{A}, \square^{\emptyset}) \mid \mathcal{A} \dashv u :: \tau[\Gamma]}$$

$$\text{EANEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau' \mid \mathcal{S}' \rightsquigarrow d \dashv \Delta \qquad \Gamma; \mathcal{C} \vdash (\!|e|\!)^u \Leftarrow \tau \mid \mathcal{A}}{\Gamma;\ \mathcal{C} \vdash (\!|e|\!)^u \Leftarrow \tau \mid \mathcal{A} \rightsquigarrow (\!|d|\!)^u_{\mathrm{id}(\Gamma)} : \tau \mid \mathrm{match}(\mathcal{A}, \square^{\emptyset}) \mid \mathcal{A} \dashv u :: \tau[\Gamma]}$$

Figure C.3: Elaboration judgements

The inserted cast on d_1 in ESApp goes from the actual synthesised type for d_1 (which could be ?) to it's corresponding function type. Both the analysis part requiring it to be *a function* (any function) and the parts specifying its component types if synthesised are retained. Analysis elaboration produces both synthesis and analysis slices as EAFun must retain the argument context slice to explain unannotated functions, while the synthesis part is determining unannotated functions' output type.

C.10. Dynamics

Final forms, contextual dynamic, and hole substitution remain the same as the Hazel calculus. But the cast transitions need to work with cast slices. The only difference is that instruction transitions require the use of slice ground matching.

$\boxed{S \blacktriangleright_{\text{ground}} \mathcal{S}'}$ $\mathcal{S}$ matches ground slices $\mathcal{S}'$

$$\frac{\tau_1 \to \tau_2 \neq ? \to ?}{\begin{array}{c}\tau_1 \to \tau_2 \mid p_1 * \mathcal{S}_1 \to p_2 * \mathcal{S}_2 \mid p_1' * \mathcal{A}_1 \to p_2' * \mathcal{A}_2 \\ \blacktriangleright_{\text{ground}}\ ? \to ? \mid p_1 * \square^{\emptyset} \to p_2 * \square^{\emptyset} \mid p_1' * \text{struct}([\![\mathcal{A}_1]\!]) \to p_2' * \text{struct}([\![\mathcal{A}_2]\!])\end{array}}$$

Where struct(p) takes the *purely structural* context slice of p.

Figure C.4: Ground type matching for slices

Future Work: When type-indexed synthesis slices are converted to type-indexed analysis slices, they lose their incremental structure. A way to retain this would be beneficial. This could be done by pairing synthesis slices with a global two-hole context.

Hazel Bugs: Unboxing

When a final form (section 2.1.2) has a type, Hazel often needs to extract parts of the term according to the type during evaluation.

For example, if a term is a final form of type list, then it could be either:

- A list literal: `[1,2,3]`.
- A list with casts wrapped around it: `[1,2,3]⟨[Int]⇒[?]⟩`.
- A list cons with indeterminate tail: `1::2::?`.

Additionally, when the input is not a list at all, it returns `DoesNotMatch`, used in pattern matching. Unboxing makes use of GADTs to allow for varying output type depending on the type that the final form is being unboxed upon.

Hazel Unboxing Bugs

While writing the search procedure I found various unboxing bugs in Hazel. Programs exhibiting these were removed from the evaluation data, whereas some bugs were fixed by myself (hence, not removing from the evaluation data).

For example, there was the following bug, affecting pattern matching. A list cons which has an indeterminate tail would *indeterminately match* with *any* list literal pattern (of *any* length), even when it is known for certain that it could never match. For example a list cons `1::2::?` represents lists with length ≥ 2, but even when matching a list literal of length 0 or 1 it would indeterminately match rather than explicitly *not* match.

Pattern matching checks if each pattern matches the scrutinee with the following behaviour, starting from the first branch:

- *Branch matches?* Execute the branch.
- *Branch does not match?* Try the next branch.
- *Branch indeterminately matches?* Cannot assume the branch doesn't match so must stop evaluation here. The match statement is then indeterminate.

Figure D.1 demonstrates a concrete example which would get stuck in Hazel, but does *not* need to. I reported and fixed this, with my PR merged into the dev branch.

```
case 1::?
  | [] => 0
  | x::xs => x
end
```

Figure D.1: Pattern Matching Bug

Extended Pattern Matching Instantiation

This appendix contains notes on instantiating holes in a match expression according also to the structure of the patterns in a match expression.

To implement this, the scrutinee needs to be matched against a pattern *and* instantiated at the same time. Crucially, instead of just giving up during indeterminate matches, the indeterminate part can be instantiated until it matches.

However, this is not always enough to actually *allow* destructuring using that branch in a match statement. The possibility that *more specific* patterns could be present above the current branch means the resulting instantiation might still be result in an indeterminate match. For example, the following would be an indeterminate match:

```
case ?::?  | [] => [] | x::y::[] => [] | x::xs => xs
```

Figure E.1: More Specific Matches

To account for this, we can take ideas from pattern matrix techniques for producing exhaustivity warnings, [50]. That is, we could generate a set of patterns which explicitly do not match any of the previous branches, then intersect those patterns with the current branch, and instantiate according to this intersection.

Supplementary Results and Corpus Data

Statistics on the corpus are found in fig. F.1.

	Count	**Prog. Size**		**Trace Length**	
		Avg.	Std. dev.	Avg.	Std. dev.
Unannotated	404	117	81	9	9
Annotated	294	117	76	9	9
Searched	203	120	77	10	10
(Total)	698	117	79	9	9

Figure F.1: Hazel Program Corpus

The averages for each search method over their successful programs for each implementation are given in fig. F.2. Note that each method succeeds on *differing* sets of programs.

	Averages	**Implementations**			
	unit	`DFS`	`BDFS`	`IDFS`	`BFS`
Time	ms	7.6	73	140	120
Major Heap	mB	3.7	32	5.9	25
Minor Heap	mB	66	680	1900	1300

Figure F.2: Benchmarks: Search Implementations

Cast are between slices 'from' a type 'to' a type. Their average sizes given in fig. F.3.

Averages		**Subdivisions**			
		Ok		**Errors**	
	unit	from	to	from	to
Cast Slice	size	5.5	1.2	5.9	1.5
Std. dev.		8.1	3.7	7.1	2.0
Proportion	%	1	0.2	1	0.2
Std. dev.		1	0.5	1	0.4
(Unannotated)					
Type Slice	size	4.8	6.3	6.9	4.4
Std. dev.		11	13	9.1	9.3
Proportion	%	1	2	2	1
Std. dev.		1	1	1	1
(Annotated)					

Figure F.3: Effectiveness: Cast Slices

Trace length and instantiation size data in fig. F.4.

	DFS	BDFS	BFS	IDFS
Witness Size Avg.	1.1	1.9	1.4	2
Std. dev.	1.2	2.3	1.4	2.3
Trace size Avg.	33	32	11	17
Std. dev.	35	33	2.4	5

Figure F.4: Witness & Trace Sizes

Some examples of programs in the translated corpus are given in fig. F.5. All three were examples where BDFS did not terminate, and are accompanied with their failure classification.

```
type expr =
  + VarX
  + Sine(expr)
  + Cosine(expr)
  + Average(expr, expr)
in let exprToString : forall a -> expr -> [a] = typfun a -> fun e -> case e
  | VarX => []
  | Sine(e1) => exprToString@<a>(e1)
  | Cosine(e1) => exprToString@<a>(e1)
  | Average(e1, e2) => exprToString@<a>(e1) ++ exprToString@<a>(e2)
end in ?
```

Depth-first bias caused the procedure to try mostly permutations of `Sine(...)` and `Cosine(...)`. The error was on the `Average(...)` branch, not found within the time limit.

(a) Witness Exists: prog2270.typed.hazel

```
type expr =
  + VarX
  + Times(expr, expr)
in let exprToString : expr -> String = fun e -> case e
  | VarX => "x"
  | (Times(e1), e2) => exprToString(e1) ++ " * " ++ exprToString(e2)
end in ?
```

A tuple pattern is used when an `expr` is expected. Instantiation only tries value of type `expr`. Further, another error exists inside this inaccessible branch.

(b) Dead Code – Wildcard: prog0080.typed.hazel

```
type expr =
  + VarX
  + Sine(expr)
  + Average(expr, expr)
  + MyExpr(expr, expr, expr, expr)
in let exprToString : expr -> String = fun e -> case e
  | VarX => "x"
  | Sine(m) => "sin(pi*" ++ exprToString(m) ++ ")"
  | Average(m, n) =>
  "((" ++ exprToString(m) ++ "+" ++ exprToString(n) ++ ")/2)"
  | MyExpr(m, n, o, p) => ?
end in let _ = exprToString(MyExpr((VarX, ?, VarX))) in ?
```

Product arity inconsistency is present in inaccessible code bound to the wildcard pattern.

(c) Dead Code – Pattern Cast Failure: prog0339.typed.hazel

Figure F.5: (Paraphrased) Failure Examples

Unimplemented Usability Improvements & Extensions

Below are proposed improvements, which the architecture of both Hazel and the new features can easily support:

- Display type slices only *upon request* using the Hazel context inspector, showing the *analysing* and *synthesising* types of the selected expression, see fig. 4.10.
- Allow users to deconstruct type slices to query specific parts, e.g. select the just the sub-slice explaining the *return type* of a function or just the function arrow part. This could be done by selecting the type parts in the context inspector. This interaction would help users really understand how code comes together to define it's types.
- Visualise graphs of cast dependencies, showing the *execution* context leading to a cast error, summarising more concisely than full evaluation traces.
- Provide a UI for the search procedure's execution traces and instantiations, integrated with Hazel's trace visualiser. This could include trace compression for better readability (e.g., skipping irrelevant function calls).
- Implement key bindings to cycle through indeterminate evaluation paths more quickly.

On Representing Non-Determinism

High Level Representation

Other high level representations, besides monads, include:

Logic Programming DSL: Languages like Prolog [17] and Curry [9] express non-determinism by directly implementing *choice* via non-deterministic evaluation. Prolog searches via backtracking, while Curry abstracts the search procedure.

There are ways to embed this within OCaml. Inspired by Curry, Kiselyov [22] created a tagless final style [41] domain specific language within OCaml. This approach fully abstracts the search procedure from the non-deterministic algorithm constructs.

Delimited Continuations: Delimited continuations [6, 70] are a control flow construct that captures a portion of the program's execution context (up to a certain delimiter) as a first-class value, which can be resumed later (in OCaml, a function). This enables writing non-deterministic code by duplicating the continuations and running them on each possibility in a choice.

Effect Handlers: Effect handlers allow the description of effects and factors out the handling of those effects. Non-determinism can be represented by an effect consisting of the choice and fail operators [8, 37], while handlers can flexibly define the search procedure and accounting logic, e.g. storing solutions in a list. As with delimited continuations, to try multiple solutions, the continuations must be cloned.

Reasoning Against

Direct implementation: This would not allow for easily abstracting the search order and would obfuscate the workings of the indeterminate evaluation and instantiation algorithms. Some sort of high level representation is also massively beneficial for readability and understanding.

Effect handlers: Multiple continuation effect handlers were not supported by Javascript of OCaml (JSOO).[1]

Continuations: Directly writing continuations is difficult and generally more unfamiliar to OCaml developers as opposed to monadic representations.

Optimised DSL: Introducing a formal DSL including optimisations, such as the proposed Tagless-Final DSL [41, 22], is very complex. However, this would allow more flexibility in writing non-deterministic evaluation, with some optimisations made automatically by the DSL.

[1] An important dependency of Hazel.

Merges & Commits

Over the all branches, **275** commits, excluding merges, were made. Below is a list of merges performed from the Hazel main branch during development and hours taken resolving conflicts with my work. Major merges are in bold, those in red introduced bugs.

Date	Hash	Time Spent	Description
08/04/25	a8efff7cbfdc87eaab3e57d8895562aed1f83a49	0m	Formatting fixes
08/04/25	d3da061988cb0d4174a36282d9588d20d11d8ebd	15m	Updated Dependencies
08/04/25	33d80b9ae3fa7ade69bbefee1dcc10aaaebc11d1	20m	Probes & Live Projectors
08/04/25	96adcc111a650ed2d709a965adcac14e7197ccc5	**45m**	**Syntax Factory**
08/04/25	2d1ab30445ecce5ce3d0f79b2b714031fd762af0	**50m**	**Id tagging a type parameter**
08/04/25	451e6f76b56874ec3ffa02002cf6a4013193d9d9	**2h 15m**	**Pattern coverage rework**
08/04/25	117f689dda9794515b5c464d246a792e67416551	0m	Context inspector fixes
08/04/25	672f7ed7814a42b93f3e4efa4a390534c25d35ad	0m	Double parenthesisation fixes
08/04/25	971763990ead191077249c72740b3ad6a657dfda	0m	Code coverage settings
08/04/25	7404b6cfdc3c4b683aa6723532eb62ca7085e33f	0m	Bring back function names
08/04/25	5f18db680913ce8be1d795d20bd54bf2f5e89f01	10m	Unboxing fixes
08/04/25	f197b65e0fb00fbc7c86a15005cd6a7ed376ac46	0m	Code coverage support
08/04/25	fbf415ea3347971672e50665ae4616d98409f087	0m	Cast & TypFun bugfix
08/04/25	661eae7621e509d762715eeecd56888078faf9b8	10m	Make join symmetric
08/04/25	f288788ae026a18bc3cd361da6e12ee716cc8723	0m	Editor statics fixes
08/04/25	b671df993ea19171c21cd248e6857ad932baf9f3	0m	Structure editor polishing
07/04/25	d7a2e2a242a474c0e50abec8659004a7ee2ab034	**7h**	**Labelled Tuples**
06/04/25	18185d16f23d79be47e7282aae7d0b1f77c4ed88	10m	Evaluator clean-up
06/02/25	f082201f1cf606068bb037a0da1c896f585e4974	30m	Fix generalised closures
31/01/25	c6f23de39b5e64e352e4d7cbab5082cc50f3b658	10m	Add builtins & bugfixes
17/12/24	0dd744b1c28ca1c11dfb6368051633dcc56b8353	0m	Add tests
16/12/24	a145ac508b5a50b132f463b2ad670d6eaa3d3b84	0m	Stepper UI fixes
11/12/24	8b189bd667ac0f1a91a18781bb5db183c58ba29f	**5h**	**MVU UI overhaul**

Project Proposal

Description

This project will add some features to the Hazel language [2]. Hazel is a functional research language that makes use of gradual types to support unusual features such as: holes (code placeholders) to give type meaning to incomplete programs. Importantly for this project, all Hazel programs, even ill-typed or incomplete programs, are evaluable. This allows dynamic reasoning about ill-typed programs via evaluation traces with the potential to improve the user's understanding of *why ill-typed programs go wrong*. See example below:

```
let rec sum : Int -> Int = fun n ->
  if n == 0 then
    true    // Type error statically caught and correctly localised
  else
    n + sum(n - 1)
in sum(2)
```

But evaluation is still possible; see below a (compressed) trace to a stuck value exhibiting a cast error:

$$\texttt{sum(2)} \mapsto^* \texttt{2 + sum(1)} \mapsto^* \texttt{2 + (1 + true}^{\langle \texttt{Bool} \Rightarrow \texttt{Int} \rangle}\texttt{)}$$

This project aims to exploit further this potential by providing some extra features to both: aid with finding values/inputs that demonstrate why type-errors were found (type-error witnesses) and linking the evaluation traces back to source code. But is not expected to directly measure the usefulness of such evaluation traces themselves in debugging, nor is the design space for a Hazel debugger inspecting and interacting with traces to be explored.

Searching for type-error witnesses automatically is the main feature provided by this project, inspired by Seidel et al. [32]. The intended use of this is to automatically generate values (for example, function arguments) that cause ill-typed programs to 'go wrong' (lead to a cast error). More specifically, the search procedure can be thought of as evaluating a *special hole* which refines its type dynamically and non-deterministically instantiates itself to values of this type to find a value whose evaluation leads to a *general* cast error – 'general' meaning excluding trivial cast errors such as generating a value that doesn't actually have the refined expected type.

Such a search procedure is undecidable and subject to path explosion, hence the success criteria (detailed below) does not expect witnesses to be provided in general, even if they do exist. Sophisticated heuristics and methods to limit path explosion to support large code samples is not a core goal.

Formal semantics of this procedure and associated proofs is an extension goal, consisting of preservation proofs and witness generality proofs (formalising the notion of generality mentioned previously).

Secondly, *cast slicing* will track source code that contributed to any cast throughout the cast elaboration and evaluation phases. In particular, this allows a cast involved in a cast error relating to a type-error witness to point back to offending code. This is expected in some sense to be similar to blame tracking [47], error and dynamic program slicing [58, 74], although these are not directly relevant for this project.

Work required for the creation of an evaluation corpus of ill-typed hazel programs, requiring manual work or creation of automated translation and/or fuzzing tools, is timetabled.

Starting Point

Only background research and exploration has been conducted. This consists of reading the Hazel research papers [2] and various other related research topics including: gradual types, bidirectional types, symbolic evaluation, OCaml error localisation and visualisation techniques.

More research, into the Hazel codebase in particular, and concrete planning is required and is timetabled accordingly.

Success Criteria

Core goals are the minimum expected goals that must be completed to consider this project a success. This corresponds to a working tool for a large portion of Hazel.

Extension goals will be timetabled in, but are relatively more difficult and not required for the project to be considered a success.

First, I give some definitions of terms:

- **Core Calculus** – The formal semantics core of Hazel as referred to by the Hazel research papers [24].
- **Basic Hazel** – A Hazel subset consisting of the core calculus, product and sum types, type aliases, bindings, (parametric) lists, bools, int, floats, strings, and their corresponding standard operations.
- **Full Hazel** – Hazel, including **Basic Hazel** plus pattern matching, explicit impredicative system-F style polymorphism and explicitly recursive types.
- **Core Corpus** – A corpus of ill-typed Hazel programs that are similar in complexity and size to student programs being taught a functional language, *e.g. (incorrect) solutions to the ticks in FoCS*. This will include examples in **Basic** or **Full Hazel** as required.
- **Extended Corpus** – A corpus of ill-typed Hazel programs that are larger in size, more akin to real-world code.
- **Evaluation Criteria** – Conditions for the search procedure to meet upon evaluation:
  1. Must have reasonable coverage – success in finding an *existing* witness which is correct and general.
  2. Must find witnesses in an amount of time suitable for interactive debugging – in-line with build-times for a debug build of existing languages.

Core Goals

- Success criteria for Cast Slicing – Cast slicing must be *correct* (slices must include all code involved in the cast) and work for *all casts*, including casts involved in cast errors. Informal reasoning in evidence of satisfying these conditions is all that will be required.
- Success criteria for the Search Procedure – The procedure must work for **Basic Hazel**, meeting the **Evaluation Criteria** over the **Core Corpus**. Analysis of some classes of programs for which witnesses could not be generated is also expected.

Extension Goals

- Search Procedure Extensions – Support for **Full Hazel** under the same criteria as above.
- Search Procedure Performance Extensions – Meeting of the **Evaluation Criteria** over an **Extended Corpus**
- Formal Semantics – The specification of a formal evaluation semantics for the search procedure over the **Core Calculus**. Additionally, a preservation and witness generality proof should be provided.

Work Plan

21st Oct *(Proposal Deadline)* – 3rd Nov

Background research & research into the Hazel semantics, cast elaboration, type system, and codebase. Produce implementation plan for cast slicing and the search procedure for the **Core Calculus**. This includes an interaction design plan, expected to be very minimal.

Milestone 1: Plan Confirmed with Supervisors

4th Nov – 17th Nov

Complete implementation of Cast Slicing for the **Core Calculus**. Write detailed reasoning for correctness, including plan for **Basic Hazel**. Add unit testing.

*Milestone 2: Cast slicing is complete for the **Core Calculus**.*

18th Nov – 1st Dec *(End of Full Michaelmas Term)*

Complete implementation of the search procedure for the **Core Calculus**.

*Milestone 3: Search Procedure is complete for the **Core Calculus**.*

2nd Dec – 20th Dec

Extension of both cast slicing and the search procedure to **Basic Hazel**.

*Milestone 4: Cast slicing & search procedure are complete for **Basic Hazel***

21st Dec – 24th Jan *(Full Lent Term starting 16th Jan)*

Basic UI interaction for the project. Drafts of Implementation chapter. Slack time. Expecting holiday, exam revision, and module exam revision. Should time be available, the **Formal Semantics** extension will be attempted.

Milestone 5: Implementation chapter draft complete.

25th Jan – 7th Feb *(Progress Report Deadline)*

Writing of Progress Report. Planning of evaluation, primarily including decisions and design of tools to be used to collect/create the **Core Corpus** and planning the specific statistical tests to conduct on the corpus. Collected corpus and translation method will be one of:

1. Manual translation of a small ill-typed OCaml program corpus into ill-typed Hazel.
2. Manual insertion of type-errors into a well-typed Hazel corpus.
3. Collection of a well-typed Hazel corpus.
   Tools: A Hazel type fuzzer to make the corpus ill-typed.
4. Collection of a well-typed OCaml corpus.
   Tools: OCaml → Hazel translator/annotator which works with well-typed OCaml. A Hazel type fuzzer.
5. Collection of an ill-typed OCaml corpus.
   Tools: OCaml → Hazel translator which works with ill-typed OCaml. *This would NOT be expected to be an implicitly typed Hazel front-end which maintains desireable properties like parametricity.*

Milestone 6: Evaluation plan and corpus creation method confirmed with supervisors.
Milestone 7: Underlying corpus (critical resource) collected.

8th Feb – 28th Feb

Implementation of the required tools for evaluation as planned. Some existing code or tools may be re-used, such as the OCaml type-checker.

*Milestone 8: **Core Corpus** has been collected.*

1st Mar – 15th Mar *(End of Full Lent Term)*

Conducting of evaluation tests and write-up of evaluation draft including results.

Milestone 9: Evaluation results documented.
Milestone 10: Evaluation draft complete.

16th Mar – 30th Mar

Drafts of remaining dissertation chapters. If possible, collection and evaluation of **Extended Corpus** using the same tools as the **Core Corpus**.

Milestone 11: Full dissertation draft complete and sent to supervisors for feedback.

31st Mar – 13th Apr

Act upon dissertation feedback. Exam revision.
Milestone 12: Second dissertation draft complete and send to supervisors for feedback.

14th Apr – 23rd Apr *(Start of Full Easter Term)*

Act upon feedback. Final dissertation complete. Exam revision.
Milestone 13: Dissertation submitted.

24th Apr – 16th May *(Final Deadline)*

Exam revision.
Milestone 14: Source code submitted.

Resource Declaration

- Underlying Corpus of either: Well-typed OCaml programs, Ill-typed OCaml programs, Hazel programs. For use in evaluation. The required tools or manual translation to convert these into the ill-typed Hazel **Core Corpus** are detailed and allocated time in the timetable.
- Hazel source code. Openly available with MIT licence on GitHub [3].
- My personal laptop will be used for development, using GitHub for version control and backup of both code and dissertation. I accept full responsibility for this machine and I have made contingency plans to protect myself against hardware and/or software failure. A backup pc is available in case of such failure.